Spatio-temporal Pattern formation: Effect of nonlocal interactions

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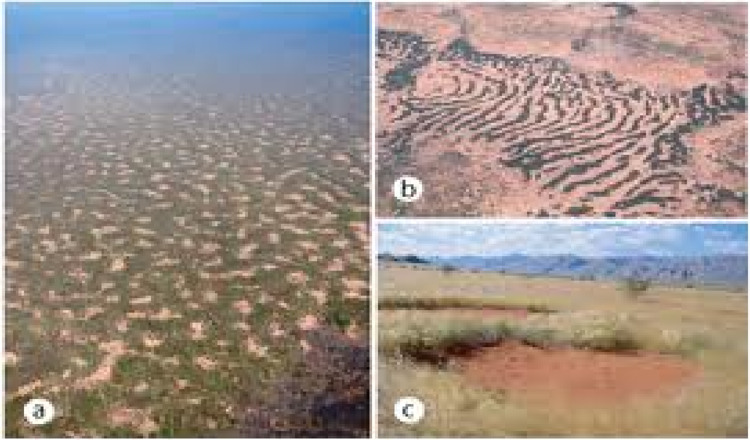
DSABNS – 2016
Evora, Portugal
02 – 05 February, 2016

Outline

Outline

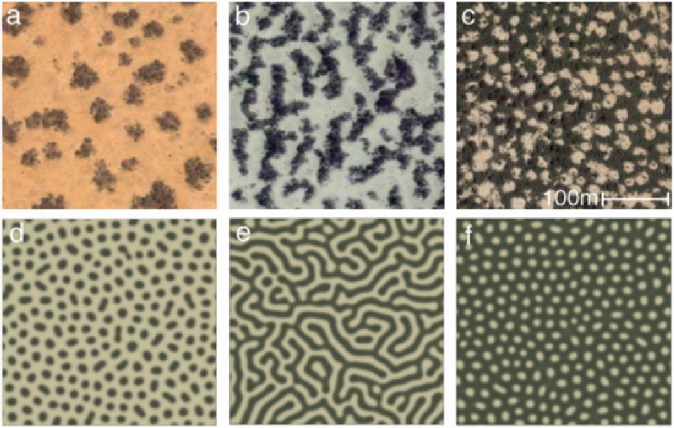
Spatio-temporal model: resulting patterns
Vegetation pattern



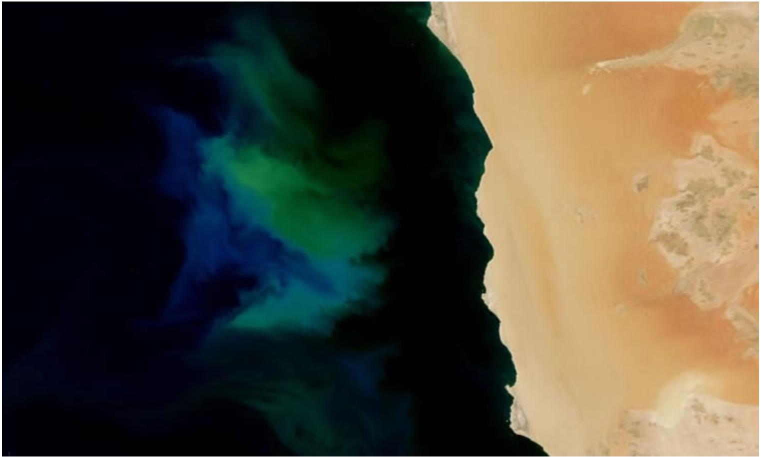
Source : http:\\ulb.ac.be/sciences/nlpc/veg_pattern.html

Spatio-temporal model: resulting patterns contd.

L. Ridolfi *et al.*, *Math. Biosci.*, **229**, 174 - 184, 2011



Spatio-temporal model: resulting patterns contd.
Marine phytoplankton bloom



Source : http:\\geology.com/nasa/marine-phytoplankton.shtml

Spatio-temporal model: resulting patterns contd.
Example of dynamic pattern, fish school

Spatio-temporal model: resulting patterns contd.
Equi-distant costal shore terraces for red crabs at Talsari, India

Spatio-temporal model: resulting patterns contd. Standard Reaction-Diffusion Model

Spatio-temporal model

$$\frac{\partial n}{\partial t} = f_1(n,p) + \left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right),$$

$$\frac{\partial p}{\partial t} = f_2(n,p) + d\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right),$$

Spatio-temporal model: resulting patterns contd.
Standard Reaction-Diffusion Model

Spatio-temporal model

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$$\frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0, \text{ on } (0,\infty) \times \partial\Omega,$$

Spatio-temporal model: resulting patterns contd.
Standard Reaction-Diffusion Model

Spatio-temporal model

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$$\frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0, \text{ on } (0,\infty) \times \partial\Omega,$$

$$n(0,x,y) = n_0(x,y) > 0,\; p(0,x,y) = p_0(x,y) > 0,$$

Spatio-temporal model: resulting patterns contd. Standard Reaction-Diffusion Model

Spatio-temporal model

$$\frac{\partial n}{\partial t} = f_1(n,p) + \left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right),$$

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$$\frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0, \text{ on } (0,\infty) \times \partial\Omega,$$

$$n(0,x,y) = n_0(x,y) > 0, \; p(0,x,y) = p_0(x,y) > 0,$$

$$(x,y) \in \Omega = [(x,y) : 0 \le x \le l, 0 \le y \le l].$$

Spatio-temporal model: resulting patterns contd. Turing bifurcation condition

$$n_t = f_1(n, p) + \nabla^2 n, \;\; p_t = f_2(n, p) + d\nabla^2 p$$

Spatio-temporal model: resulting patterns contd.
Turing bifurcation condition

$$n_t = f_1(n,p) + \nabla^2 n, \;\; p_t = f_2(n,p) + d\nabla^2 p$$

- (n_*, p_*) is temporal steady state, $f_1(n_*, p_*) = 0 = f_2(n_*, p_*)$

Spatio-temporal model: resulting patterns contd. Turing bifurcation condition

$$n_t = f_1(n,p) + \nabla^2 n, \;\; p_t = f_2(n,p) + d\nabla^2 p$$

- (n_*, p_*) is temporal steady state, $f_1(n_*, p_*) = 0 = f_2(n_*, p_*)$
- Stability of (n_*, p_*): $a_{11} + a_{22} < 0, \;\; a_{11}a_{22} - a_{12}a_{21} > 0,$

Spatio-temporal model: resulting patterns contd.
Turing bifurcation condition

$$n_t = f_1(n, p) + \nabla^2 n, \;\; p_t = f_2(n, p) + d\nabla^2 p$$

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- $a_{11} + a_{22} = 0 \rightarrow$ Hopf-bifurcation boundary

Spatio-temporal model: resulting patterns contd.
Turing bifurcation condition

$$n_t = f_1(n, p) + \nabla^2 n, \;\; p_t = f_2(n, p) + d\nabla^2 p$$

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- $a_{11} + a_{22} = 0 \rightarrow$ Hopf-bifurcation boundary
- $n(t, x, y) = n_*, \; p(t, x, y) = p_* \rightarrow$ homogeneous steady-state

Spatio-temporal model: resulting patterns contd. Turing bifurcation condition

$$n_t = f_1(n,p) + \nabla^2 n, \;\; p_t = f_2(n,p) + d\nabla^2 p$$

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- $n(t,x,y) = n_*, \; p(t,x,y) = p_* \rightarrow$ homogeneous steady-state
- Turing instability conditions:
  (i) $a_{11} + a_{22} < 0$, (ii) $a_{11}a_{22} - a_{12}a_{21} > 0$,
  (iii) $da_{11} + a_{22} > 2\sqrt{d}\sqrt{a_{11}a_{22} - a_{12}a_{21}}$

Spatio-temporal model: resulting patterns contd.
Turing bifurcation condition

$$n_t = f_1(n,p) + \nabla^2 n, \;\; p_t = f_2(n,p) + d\nabla^2 p$$

- (n_*, p_*) is temporal steady state, $f_1(n_*, p_*) = 0 = f_2(n_*, p_*)$
- Stability of (n_*, p_*): $a_{11} + a_{22} < 0, \;\; a_{11}a_{22} - a_{12}a_{21} > 0,$
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- $da_{11} + a_{22} = 2\sqrt{d}\sqrt{a_{11}a_{22} - a_{12}a_{21}} \rightarrow$ Turing bifurcation boundary

Spatio-temporal model: resulting patterns contd.

Theor Ecol (2011) 4:37–53
DOI 10.1007/s12080-010-0073-1

ORIGINAL PAPER

Self-organised spatial patterns and chaos in a ratio-dependent predator–prey system

Malay Banerjee · Sergei Petrovskii

Spatio-temporal model: resulting patterns contd.

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Ecological Complexity 21 (2015) 199–214

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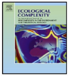

Original Research Article

Existence and non-existence of spatial patterns in a ratio-dependent predator–prey model

CrossMark

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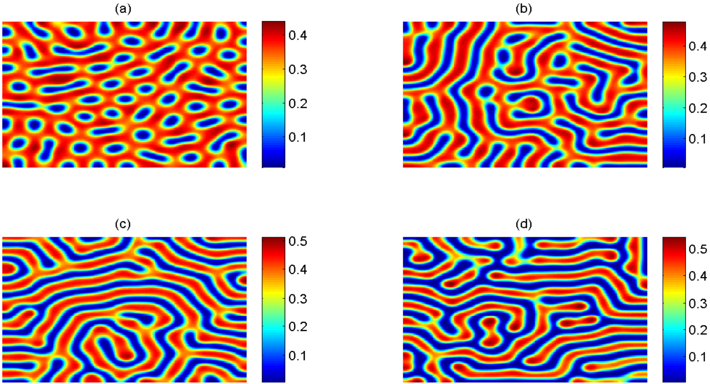
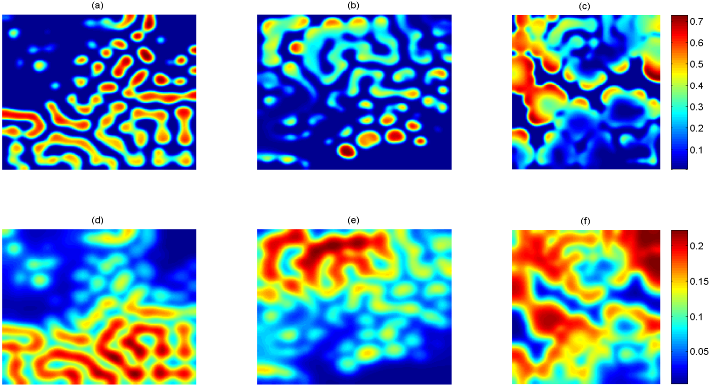
[b] School of Basic Sciences, Indian Institute of Technology Mandi, Mandi, HP 175001, India

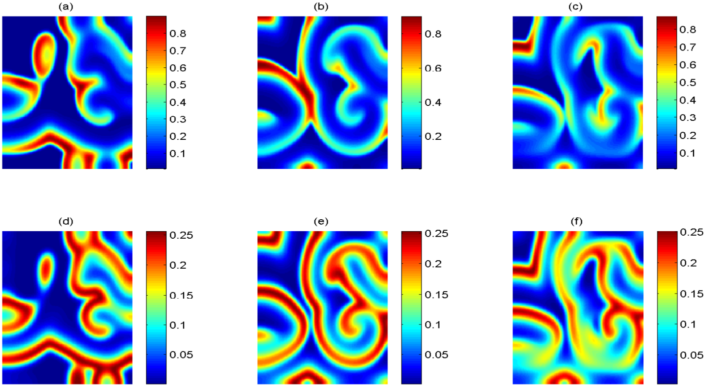
Spatio-temporal model: resulting patterns contd.
Stationary patterns



Spatio-temporal model: resulting patterns contd.
Spatio-temporal chaotic pattern (irregular patches)



Spatio-temporal model: resulting patterns contd.
Spatio-temporal chaotic pattern (interacting spiral)



Spatio-temporal model: resulting patterns contd.

Justification for considering 1D (space) model

Spatio-temporal model: resulting patterns contd.

Justification for considering 1D (space) model

Spatio-temporal model: resulting patterns contd. Justification for considering 1D (space) model

Spatio-temporal model: resulting patterns contd.
Justification for considering 1D (space) model

Spatio-temporal model: resulting patterns contd.
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Spatio-temporal model: resulting patterns contd.
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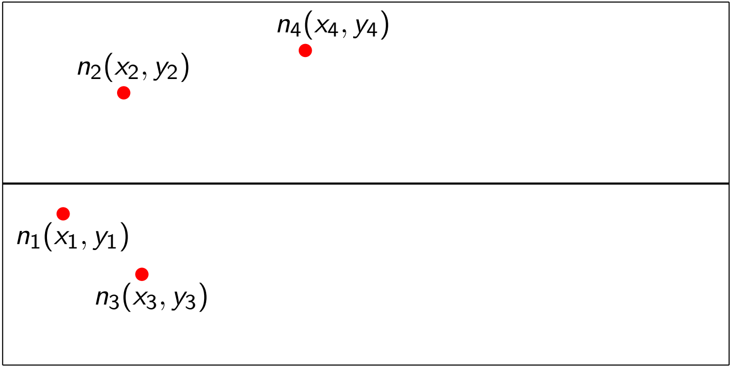
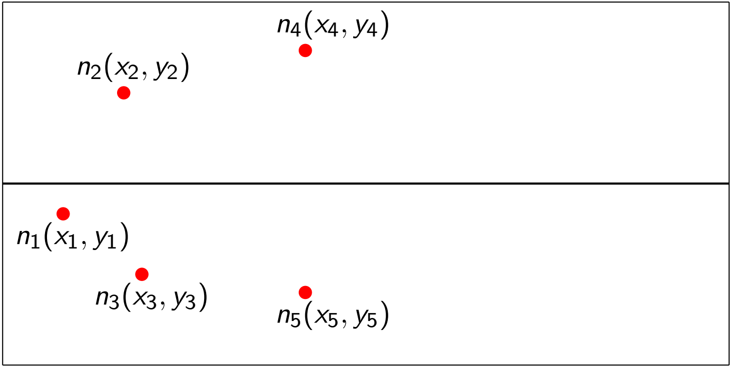
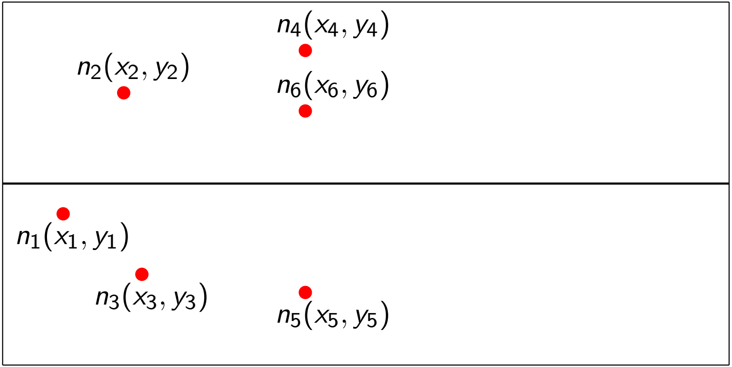


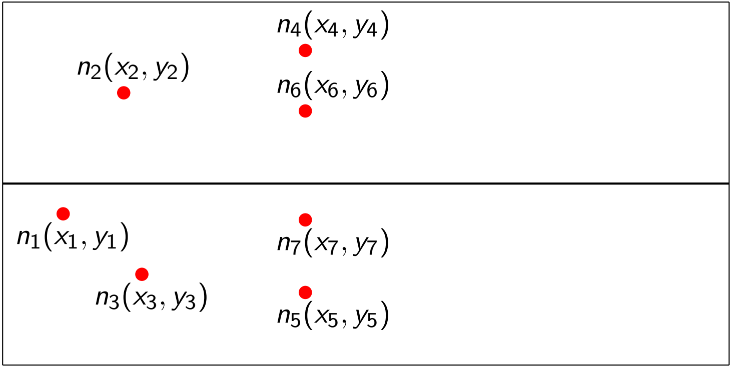
Spatio-temporal model: resulting patterns contd.
Justification for considering 1D (space) model



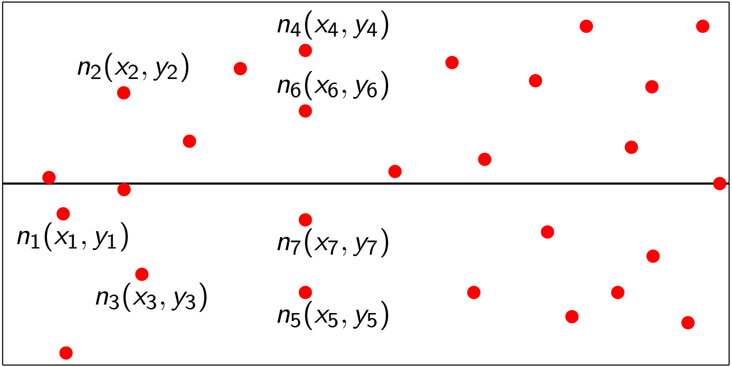
Spatio-temporal model: resulting patterns contd.
Justification for considering 1D (space) model



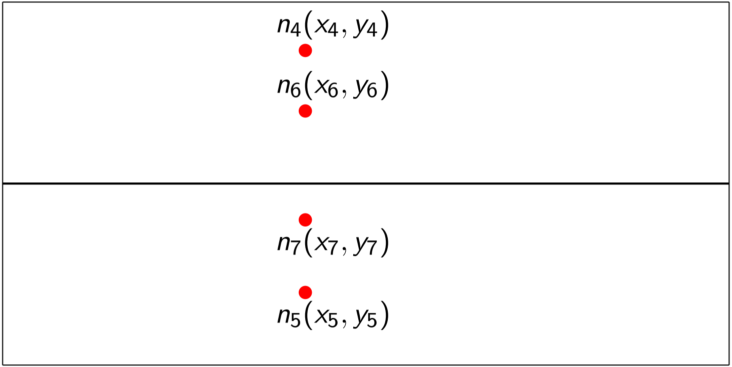
Spatio-temporal model: resulting patterns contd.
Justification for considering 1D (space) model



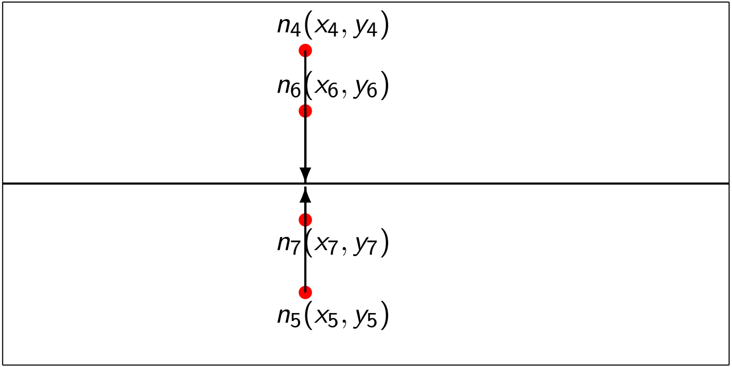
Spatio-temporal model: resulting patterns contd.
Justification for considering 1D (space) model



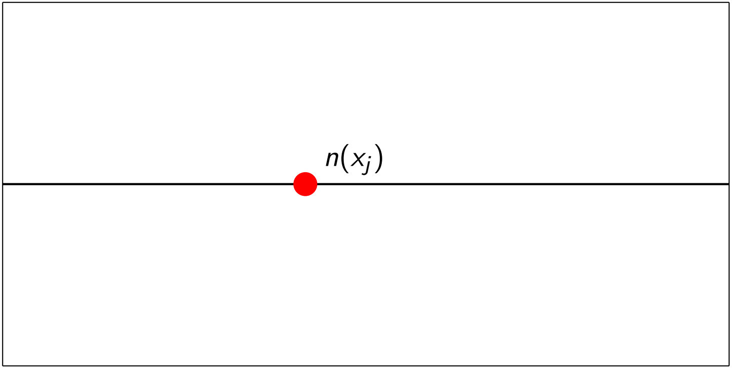
Spatio-temporal model: resulting patterns contd.
Justification for considering 1D (space) model contd.



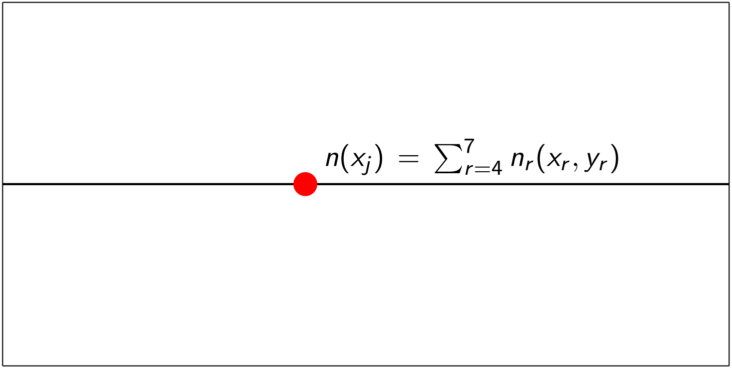
Spatio-temporal model: resulting patterns contd.
Justification for considering 1D (space) model contd.



Spatio-temporal model: resulting patterns contd.



Spatio-temporal model: resulting patterns contd.



Spatio-temporal model: resulting patterns contd.

Ratio-dependent prey-predator model

Spatio-temporal model: resulting patterns contd.

Ratio-dependent prey-predator model

$$\frac{\partial n}{\partial t} = n(1-n) - \frac{\alpha np}{n+p} + \frac{\partial^2 n}{\partial x^2}$$

$$\frac{\partial p}{\partial t} = \frac{\beta np}{n+p} - \gamma p - \delta p^2 + d\frac{\partial^2 p}{\partial x^2}$$

$$n(0,x) = n_0(x) > 0, \;\; p(0,x) = p_0(x) > 0$$

Spatio-temporal model: resulting patterns contd.

Ratio-dependent prey-predator model

$$\frac{\partial n}{\partial t} = n(1-n) - \frac{\alpha np}{n+p} + \frac{\partial^2 n}{\partial x^2}$$

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$$n(0,x) = n_0(x) > 0, \;\; p(0,x) = p_0(x) > 0$$

$$n(t,0) = n(t,l), \;\; p(t,0) = p(t,l)$$

Spatio-temporal model: resulting patterns contd.

Ratio-dependent prey-predator model

$$\frac{\partial n}{\partial t} = n(1-n) - \frac{\alpha np}{n+p} + \frac{\partial^2 n}{\partial x^2}$$

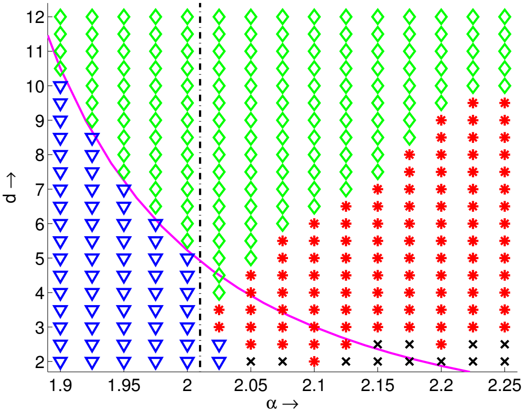
$$\frac{\partial p}{\partial t} = \frac{\beta np}{n+p} - \gamma p - \delta p^2 + d\frac{\partial^2 p}{\partial x^2}$$

$$n(0,x) = n_0(x) > 0, \;\; p(0,x) = p_0(x) > 0$$

$$n(t,0) = n(t,l), \;\; p(t,0) = p(t,l)$$

α, β, γ, δ and d are dimensionless positive parameters.

Standard Reaction-Diffusion Models contd.
Bifurcation diagram, stationary and non-stationary patterns



∇ → homogeneous stead-state, ◇ → stationary pattern, $*$ → spatio-temporal chaos, **x** → extinction. ($\beta = 1$, $\gamma = 0.6$, $\delta = 0.1$)

Spatio-temporal model: resulting patterns contd.

Oscillations and chaos behind predator-prey invasion: mathematical artifact or ecological reality?

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[1] *Nonlinear Systems Laboratory, Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK*
[2] *Department of Mathematics, University of Utah, Salt Lake City, Utah 84112, USA*

Phil. Trans. Royal Soc. B, **357**, 21 – 38 (1997)

Spatio-temporal model: resulting patterns contd.

PERGAMON

Mathematical and Computer Modelling 29 (1999) 49–63

MATHEMATICAL AND COMPUTER MODELLING

A Minimal Model of Pattern Formation in a Prey-Predator System

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Spatio-temporal model: resulting patterns contd.
RM model with PBC

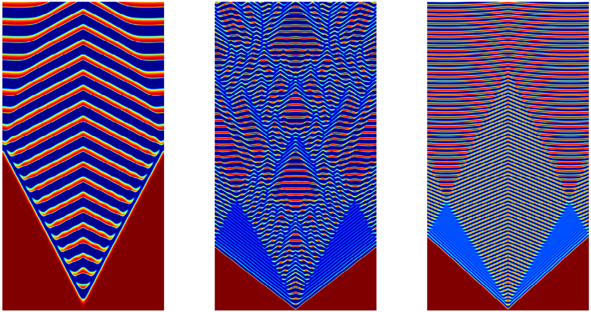
RM Model

$$n_t = n - n^2 - \frac{np}{c+n} + n_{xx}, \;\; p_t = \frac{anp}{c+n} - bp + p_{xx},$$

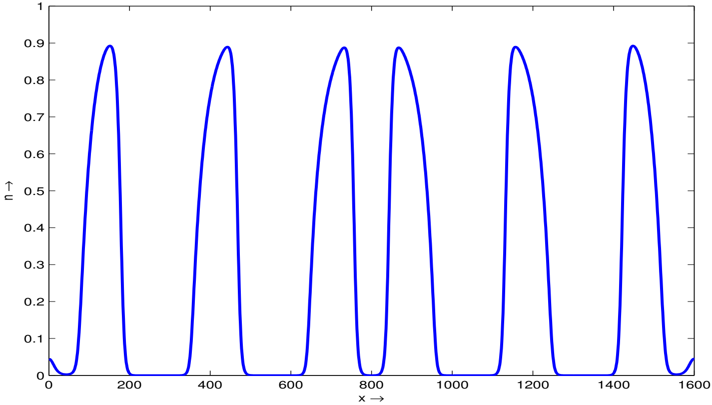
Spatio-temporal model: resulting patterns contd. RM model with PBC

RM Model

$$n_t = n - n^2 - \frac{np}{c+n} + n_{xx}, \quad p_t = \frac{anp}{c+n} - bp + p_{xx},$$

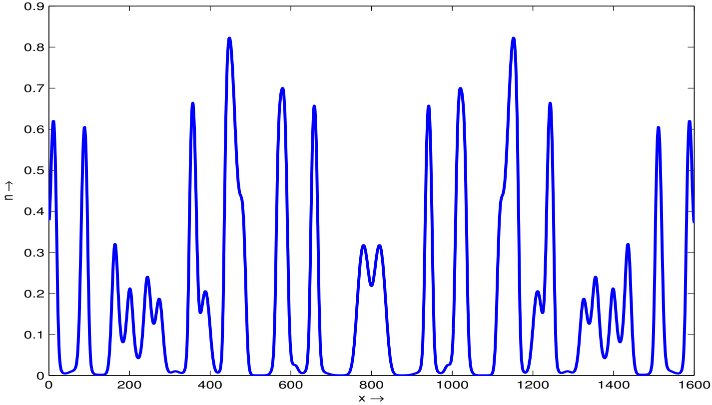


Spatio-temporal model: resulting patterns contd. RM model with PBC contd.



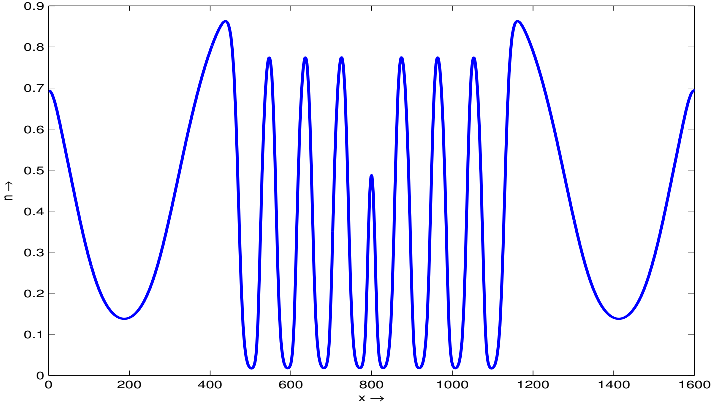
$a = 0.15$, $b = 0.05$, $c = 0.2 \rightarrow$ Periodic travelling wave

Spatio-temporal model: resulting patterns contd.
RM model with PBC contd.



$a = 0.9,\ b = 0.3,\ c = 0.2 \rightarrow$ Spatio-temporal chaos

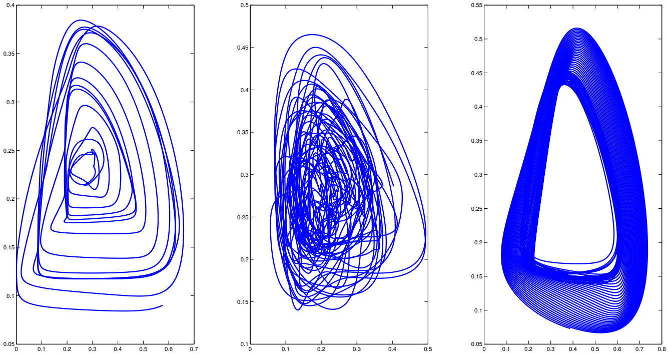
Spatio-temporal model: resulting patterns contd. RM model with PBC contd.



$a = 1.0,\ b = 0.5,\ c = 0.2 \rightarrow$ Periodic travelling wave behind the invasive wave

Spatio-temporal model: resulting patterns contd. RM model with PBC contd.

Plot of $\langle n \rangle$ against $\langle p \rangle$ after discarding initial transients.



$a = 0.15$, $b = 0.05$, $c = 0.2$ (left); $a = 0.9$, $b = 0.3$, $c = 0.2$ (middle); $a = 1.0$, $b = 0.5$, $c = 0.2$ (right)

Outline

Nonlocal reaction-diffusion model

Basic assumptions behind the models with 1D space

Nonlocal reaction-diffusion model
Basic assumptions behind the models with 1D space

1D reaction-diffusion model

$$\frac{\partial n(t,x)}{\partial t} = f_1(n(t,x),p(t,x)) + \frac{\partial^2 n(t,x)}{\partial x^2},$$
$$\frac{\partial p(t,x)}{\partial t} = f_2(n(t,x),p(t,x)) + d\frac{\partial^2 p(t,x)}{\partial x^2},$$

Nonlocal reaction-diffusion model
Basic assumptions behind the models with 1D space

1D reaction-diffusion model

$$\frac{\partial n(t,x)}{\partial t} = f_1(n(t,x),p(t,x)) + \frac{\partial^2 n(t,x)}{\partial x^2},$$
$$\frac{\partial p(t,x)}{\partial t} = f_2(n(t,x),p(t,x)) + d\frac{\partial^2 p(t,x)}{\partial x^2},$$

$$n(t-\Delta t, x) \longrightarrow n(t,x)$$

Nonlocal reaction-diffusion model
Basic assumptions behind the models with 1D space

1D reaction-diffusion model

$$
\begin{aligned}
\frac{\partial n(t,x)}{\partial t} &= f_1(n(t,x),p(t,x)) + \frac{\partial^2 n(t,x)}{\partial x^2},\\
\frac{\partial p(t,x)}{\partial t} &= f_2(n(t,x),p(t,x)) + d\frac{\partial^2 p(t,x)}{\partial x^2},
\end{aligned}
$$

$$n(t-\Delta t,x) \longrightarrow n(t,x)$$

$$n(t-\Delta t,x),\ p(t-\Delta t,x) \longrightarrow n(t,x),\ p(t,x)$$

Nonlocal reaction-diffusion model
Basic assumptions behind the models with 1D space

1D reaction-diffusion model

$$\frac{\partial n(t,x)}{\partial t} = f_1(n(t,x), p(t,x)) + \frac{\partial^2 n(t,x)}{\partial x^2},$$
$$\frac{\partial p(t,x)}{\partial t} = f_2(n(t,x), p(t,x)) + d\frac{\partial^2 p(t,x)}{\partial x^2},$$

$$n(t-\Delta t, x) \longrightarrow n(t,x)$$

$$n(t-\Delta t, x),\ p(t-\Delta t, x) \longrightarrow n(t,x),\ p(t,x)$$

$$n(t-\Delta t, x+\lambda_j \Delta x) \longrightarrow n(t,x)$$
$$p(t-\Delta t, x+\lambda_j \Delta x) \longrightarrow p(t,x)$$

Nonlocal reaction-diffusion model
Basic assumptions behind the models with 1D space

1D reaction-diffusion model

$$\frac{\partial n(t,x)}{\partial t} = f_1(n(t,x), p(t,x)) + \frac{\partial^2 n(t,x)}{\partial x^2},$$
$$\frac{\partial p(t,x)}{\partial t} = f_2(n(t,x), p(t,x)) + d\frac{\partial^2 p(t,x)}{\partial x^2},$$

$$n(t-\Delta t, x) \longrightarrow n(t,x)$$

$$n(t-\Delta t, x),\ p(t-\Delta t, x) \longrightarrow n(t,x),\ p(t,x)$$

$$n(t-\Delta t, x+\lambda_j \Delta x) \longrightarrow n(t,x)$$
$$p(t-\Delta t, x+\lambda_j \Delta x) \longrightarrow p(t,x)$$
$$\lambda_j = -1,\ 0,\ 1$$

Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

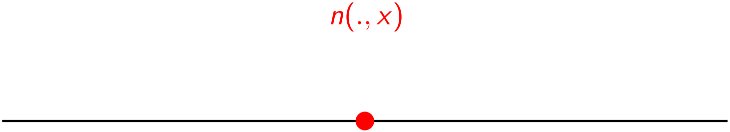
Nonlocal intra-specific competition term

Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term

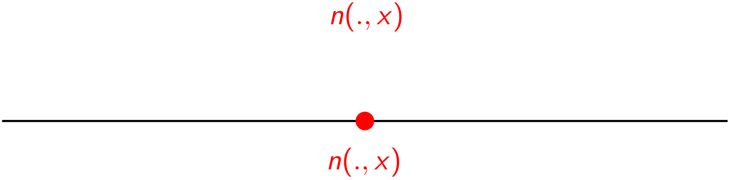
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



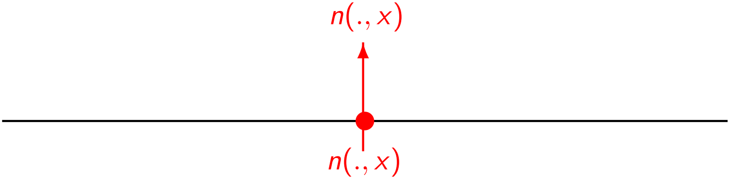
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



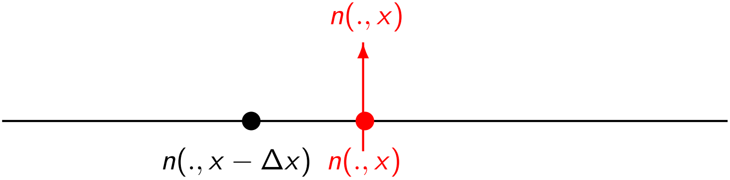
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



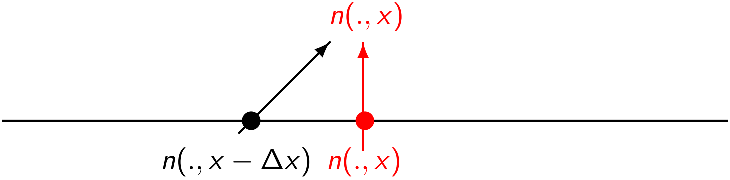
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



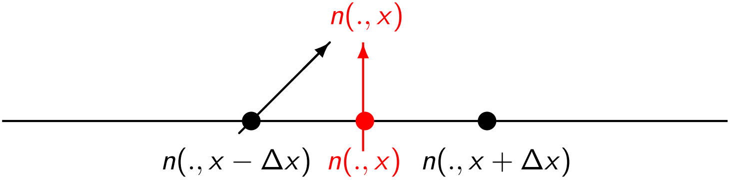
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



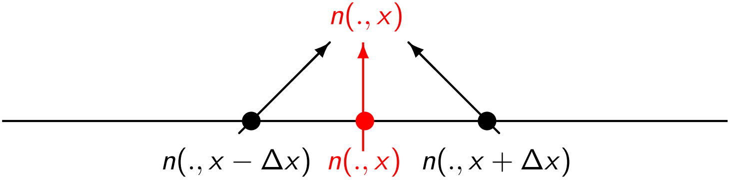
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



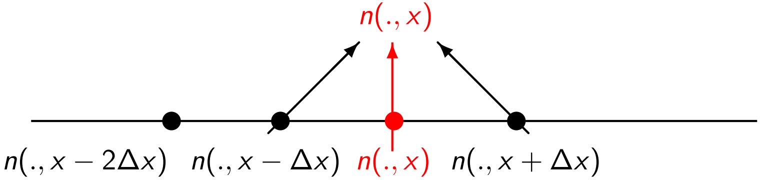
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



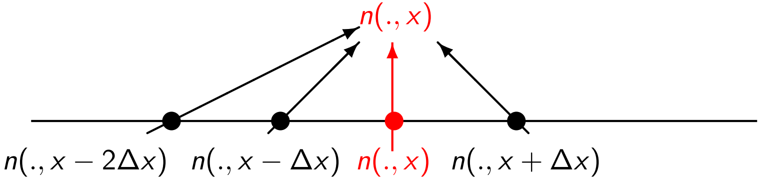
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



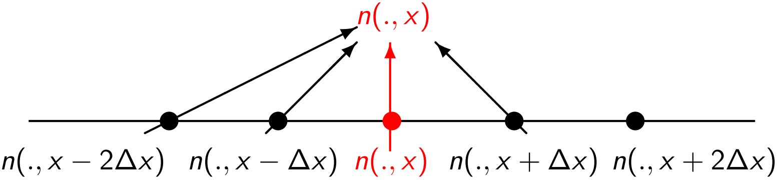
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



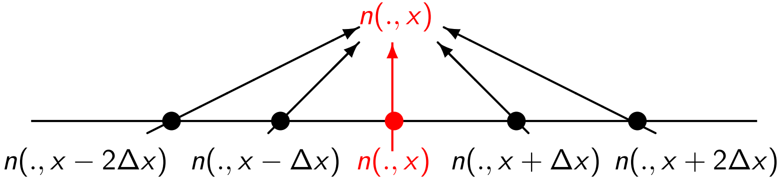
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



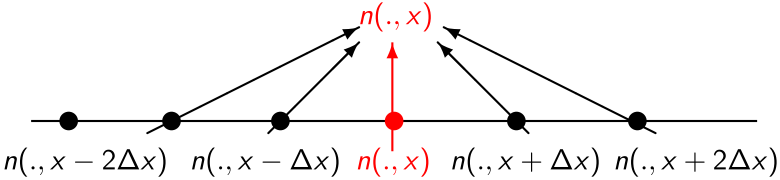
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



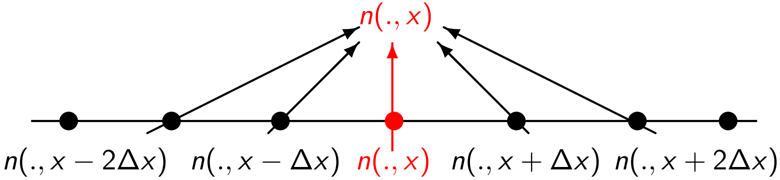
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



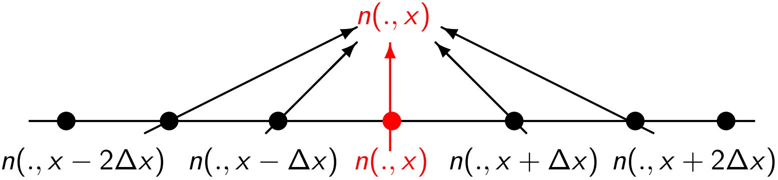
Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

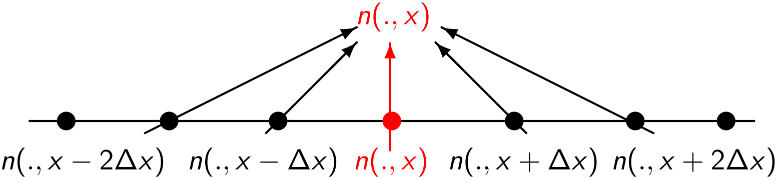
Nonlocal intra-specific competition term



$$n^2(t,x) \rightarrow n(t,x)\int_{-\infty}^{\infty}\phi(x-y)n(t,y)dy$$

Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term

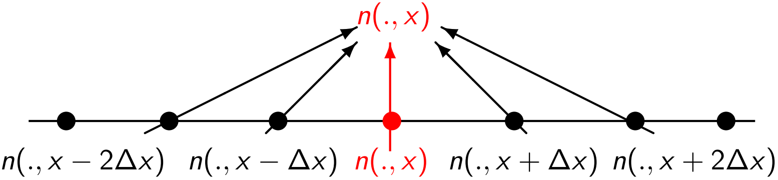


$$n^2(t,x) \to n(t,x)\int_{-\infty}^{\infty}\phi(x-y)n(t,y)dy$$

$$\frac{\partial n(t,x)}{\partial t} = n(t,x)(1 - w_1(t,x)) + \frac{\partial^2 n(t,x)}{\partial x^2}$$

Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



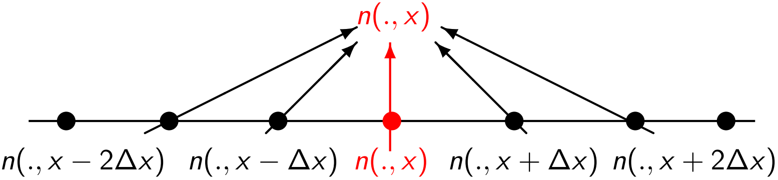
$$n^2(t,x) \rightarrow n(t,x)\int_{-\infty}^{\infty}\phi(x-y)n(t,y)dy$$

$$\frac{\partial n(t,x)}{\partial t} = n(t,x)(1 - w_1(t,x)) + \frac{\partial^2 n(t,x)}{\partial x^2}$$

$$w_1(t,x) = \int_{-\infty}^{\infty}\phi(x-y)n(t,y)dy$$

Nonlocal reaction-diffusion model contd.
Nonlocal consumption of resources

Nonlocal intra-specific competition term



$$n^2(t,x) \;\rightarrow\; n(t,x)\int_{-\infty}^{\infty}\phi(x-y)n(t,y)dy$$

$$\frac{\partial n(t,x)}{\partial t} = n(t,x)(1-w_1(t,x)) + \frac{\partial^2 n(t,x)}{\partial x^2}$$

$$w_1(t,x) \;=\; \int_{-\infty}^{\infty}\phi(x-y)n(t,y)dy$$

$\phi \rightarrow$ function with bounded support, $\int_{-\infty}^{\infty}\phi(s)ds \;=\; 1$

Nonlocal reaction-diffusion model contd.

Model with nonlocal intra-specific competition

$$\frac{\partial n(t,x)}{\partial t} = n(t,x)(1 - w_1(t,x)) - \frac{\alpha n(t,x)p(t,x)}{n(t,x)+p(t,x)} + \frac{\partial^2 n(t,x)}{\partial x^2}$$

$$\frac{\partial p(t,x)}{\partial t} = \frac{\beta n(t,x)p(t,x)}{n(t,x)+p(t,x)} - (\gamma + \delta w_2(t,x))p(t,x) + d\frac{\partial^2 n(t,x)}{\partial x^2}$$

$$w_1(t,x) = \int_{-\infty}^{\infty} \phi_1(x-y)n(t,y)dy$$

$$w_2(t,x) = \int_{-\infty}^{\infty} \phi_2(x-y)p(t,y)dy$$

For simplicity, here we assume $\phi_1(.) = \phi_2(.) \equiv \phi(.)$

Nonlocal reaction-diffusion model contd.

Instability condition

Linear eigenvalue problem:

$$\lambda u(x) = (a_1 + n_*)u(x) - n_* \int_{-\infty}^{\infty} \phi(x-y)u(y)dy + a_2 v(x) + u''(x)$$

$$\lambda v(x) = b_1 u(x) + (b_2 + \delta p_*)v(x) - \delta p_* \int_{-\infty}^{\infty} \phi(x-y)v(y)dy + dv''(x)$$

$$\left.\frac{\partial f_1}{\partial n}\right|_{(n_*,p_*)} = a_1, \quad \left.\frac{\partial f_1}{\partial p}\right|_{(n_*,p_*)} = a_2,$$

$$\left.\frac{\partial f_2}{\partial n}\right|_{(n_*,p_*)} = b_1, \quad \left.\frac{\partial f_2}{\partial p}\right|_{(n_*,p_*)} = b_2.$$

Nonlocal reaction-diffusion model contd.

Instability condition contd.

Taking Fourier transform, we get:

$$\lambda\overline{u}(\xi) = (a_1 + n_*)\overline{u}(\xi) - n_*\overline{\phi}(\xi)\overline{u}(\xi) + a_2\overline{v}(\xi) - \xi^2\overline{u}(\xi)$$
$$\lambda\overline{v}(\xi) = b_1\overline{u}(\xi) + (b_2 + \delta p_*)\overline{v}(\xi) - \delta p_*\overline{\phi}(\xi)\overline{u}(\xi) - d\xi^2\overline{v}(\xi)$$

For stability, we need:

$$a_1 + b_2 + n_* + \delta p_* - (1 + d)\xi^2 - (n_* + \delta p_*)\overline{\phi}(\xi) < 0$$

$$[-(a_1 + n_*) + n_*\overline{\phi}(\xi) + \xi^2][-(b_2 + \delta p_*) + \delta p_*\overline{\phi}(\xi) + d\xi^2] - a_2 b_1 > 0$$

Non-local reaction-diffusion model contd.

Choices for $\phi(y)$
(Physica D, **253**, 12–23, 2013)

Non-local reaction-diffusion model contd.

Choices for $\phi(y)$
(Physica D, **253**, 12–23, 2013)

$$\phi_{step}(y) = \begin{cases} \frac{1}{2M}, & |y| < M \\ 0, & |y| > M \end{cases}$$

Non-local reaction-diffusion model contd.

Choices for $\phi(y)$
(Physica D, **253**, 12–23, 2013)

$$\phi_{step}(y) = \begin{cases} \frac{1}{2M}, & |y| < M \\ 0, & |y| > M \end{cases}$$

$$\phi_p(y) = \begin{cases} \frac{3}{4M^3}(M^2 - y^2), & |y| < M \\ 0, & |y| > M \end{cases}$$

Non-local reaction-diffusion model contd.

Choices for $\phi(y)$
(Physica D, **253**, 12–23, 2013)

$$\phi_{step}(y) = \begin{cases} \frac{1}{2M}, & |y| < M \\ 0, & |y| > M \end{cases}$$

$$\phi_p(y) = \begin{cases} \frac{3}{4M^3}(M^2 - y^2), & |y| < M \\ 0, & |y| > M \end{cases}$$

$\phi(y) \equiv \phi_{step}(y)$

$$\overline{\phi}(\xi) = \frac{1}{2M}\int_{-M}^{M} \cos(\xi y)dy = \frac{\sin(\xi M)}{(\xi M)}$$

Nonlocal reaction-diffusion model contd.

Instability condition contd.

Substituting, $\overline{\phi}(\xi) = \frac{\sin(\xi M)}{(\xi M)}$ in LHS of second inequality,

$$D(\xi, M) = d\xi^4 + \left[(n_* + \delta p_*)\frac{\sin(\xi M)}{\xi M} - (a_1 + b_2 + n_* + \delta p_*)\right]\xi^2$$
$$+ \left[-(a_1 + n_*) + n_* \frac{\sin(\xi M)}{\xi M}\right]\left[-(b_2 + \delta p_*) + \delta p_* \frac{\sin(\xi M)}{\xi M}\right] - a_2 b_1$$

Nonlocal reaction-diffusion model contd.

Instability condition contd.

Substituting, $\overline{\phi}(\xi) = \frac{\sin(\xi M)}{(\xi M)}$ in LHS of second inequality,

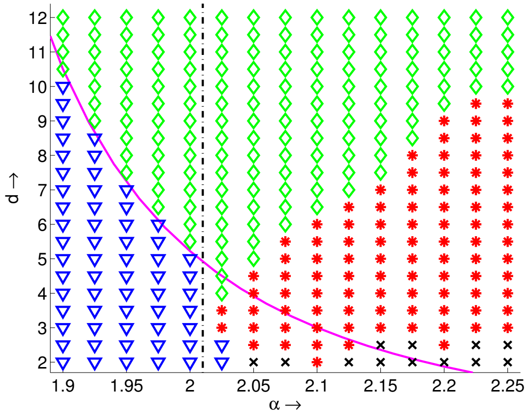
$$D(\xi, M) = d\xi^4 + \left[(n_* + \delta p_*)\frac{\sin(\xi M)}{\xi M} - (a_1 + b_2 + n_* + \delta p_*)\right]\xi^2$$
$$+\left[-(a_1 + n_*) + n_*\frac{\sin(\xi M)}{\xi M}\right]\left[-(b_2 + \delta p_*) + \delta p_*\frac{\sin(\xi M)}{\xi M}\right] - a_2 b_1$$

Turing instability condition

Pair of values of ξ and M satisfying,

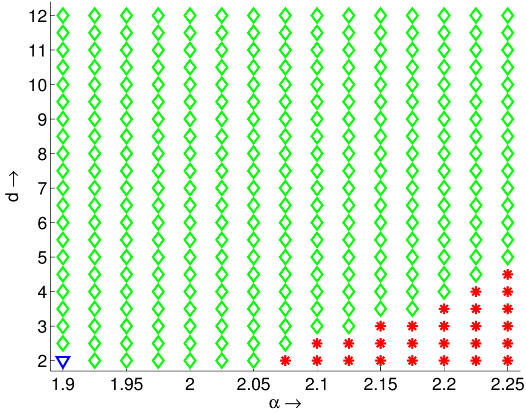
$$D(\xi, M) = 0, \quad \frac{\partial D(\xi, M)}{\partial \xi} = 0, \quad \frac{\partial D(\xi, M)}{\partial M} = 0,$$

Nonlocal reaction-diffusion model contd.
Bifurcation diagram, without nonlocal interaction



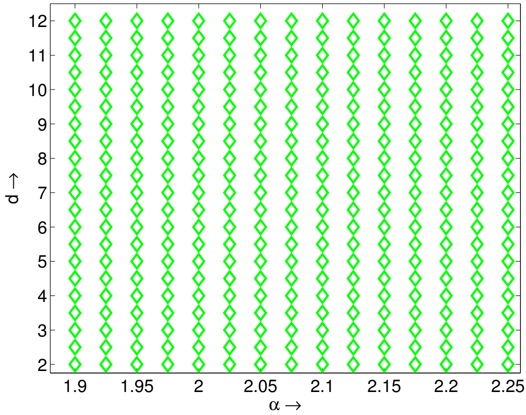
$$\beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ M = 0$$

Nonlocal reaction-diffusion model contd.
Bifurcation diagram for nonlocal model



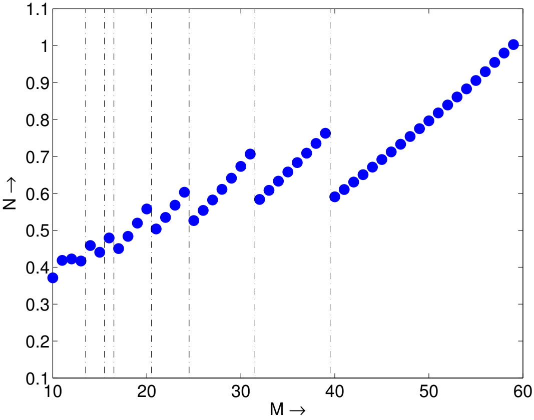
$$\beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ M = 5$$

Nonlocal reaction-diffusion model contd.
Bifurcation diagram for nonlocal model contd.



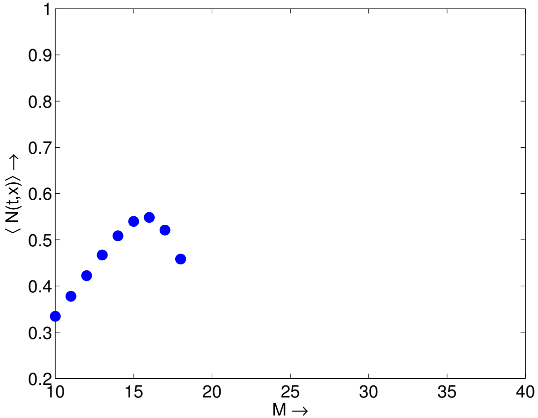
$$\beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ M = 10$$

Nonlocal reaction-diffusion model contd.
Bifurcation of steady-states with M



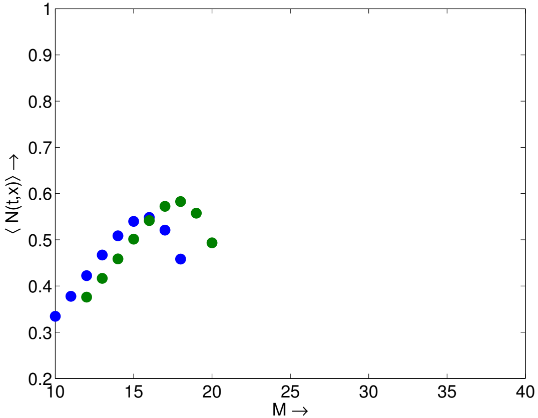
$$\alpha = 2.15,\ \beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ d = 2$$

Nonlocal reaction-diffusion model contd.
Existence of multiple steady-states (numerical continuation)



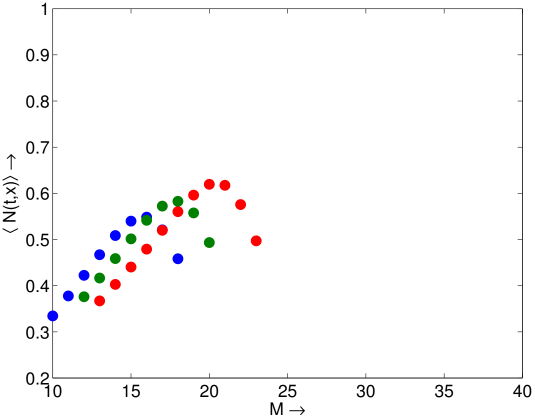
$$\alpha = 2.15,\ \beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ d = 2$$

Nonlocal reaction-diffusion model contd.
Existence of multiple steady-states (numerical continuation)



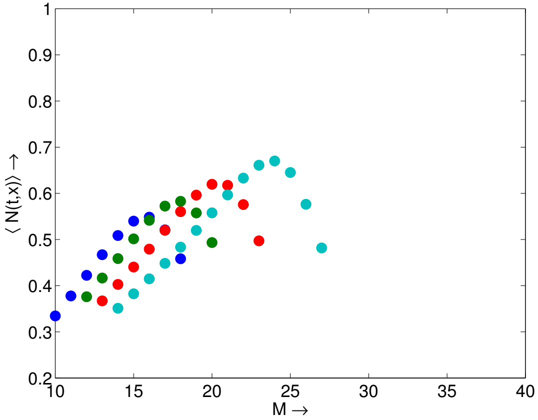
$\alpha = 2.15,\ \beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ d = 2$

Nonlocal reaction-diffusion model contd.
Existence of multiple steady-states (numerical continuation)



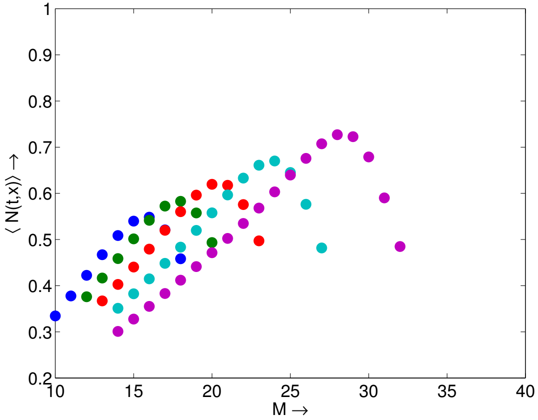
$$\alpha = 2.15,\ \beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ d = 2$$

Nonlocal reaction-diffusion model contd.
Existence of multiple steady-states (numerical continuation)



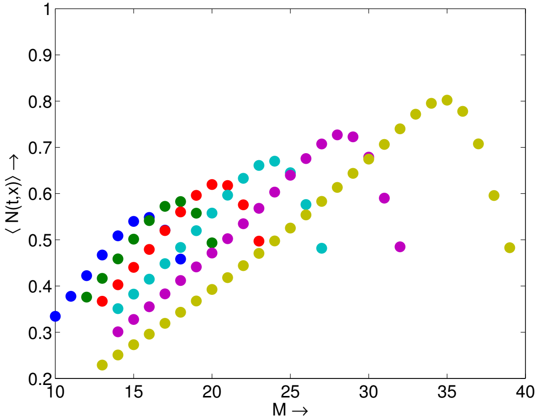
$$\alpha = 2.15,\ \beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ d = 2$$

Nonlocal reaction-diffusion model contd.
Existence of multiple steady-states (numerical continuation)



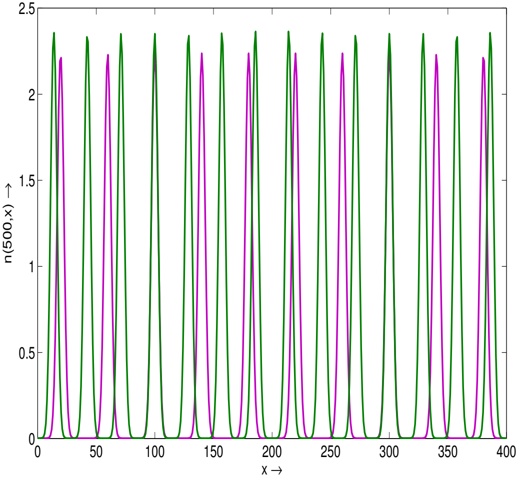
$$\alpha = 2.15,\ \beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ d = 2$$

Nonlocal reaction-diffusion model contd.
Existence of multiple steady-states (numerical continuation)

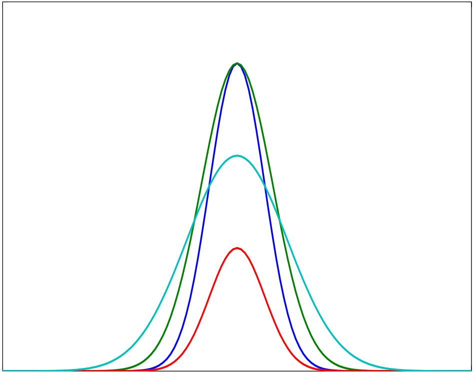


$$\alpha = 2.15,\ \beta = 1,\ \gamma = 0.6,\ \delta = 0.1,\ d = 2$$

Nonlocal reaction-diffusion model contd.
Existence of multiple steady-states (numerical continuation)



Nonlocal reaction-diffusion model contd.
Change in patch size due to variation in choice of kernel



Change in shape of the patches with different kernel functions having bounded support.

Nonlocal reaction-diffusion model contd.
RM model with nonlocal interaction

RM model with nonlocal intra-specific competition

$$\frac{\partial n(t,x)}{\partial t} = n(t,x)(1 - w(t,x)) - \frac{n(t,x)p(t,x)}{c + n(t,x)} + \frac{\partial^2 n(t,x)}{\partial x^2}$$

$$\frac{\partial p(t,x)}{\partial t} = \frac{an(t,x)p(t,x)}{c + n(t,x)} - bp(t,x) + \frac{\partial^2 p(t,x)}{\partial x^2}$$

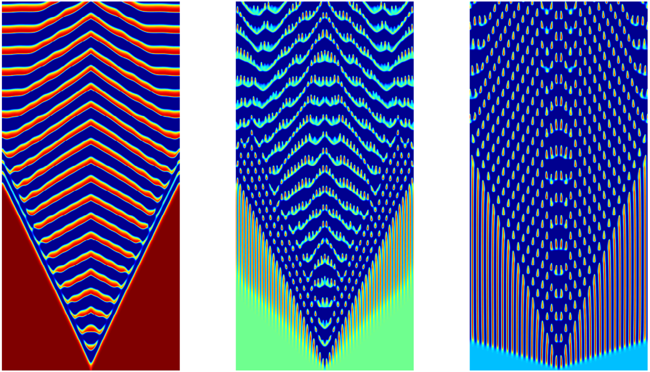
$$n(0,x) = n_0(x), \; p(0,x) = p_0(x)$$

$$n(t,0) = n(t,L), \; p(t,0) = p(t,L)$$

$$w(t,x) = \int_{-\infty}^{\infty} \phi(x-y) n(t,y) dy$$

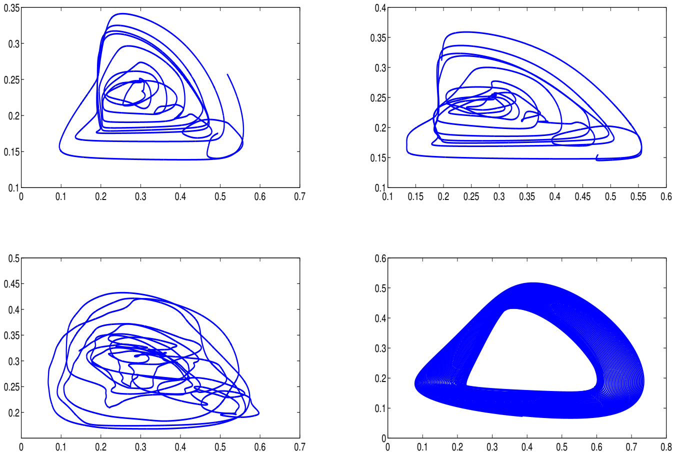
$$\phi(z) = \phi_{step}(z)$$

Nonlocal reaction-diffusion model contd.
RM model with nonlocal interaction contd.

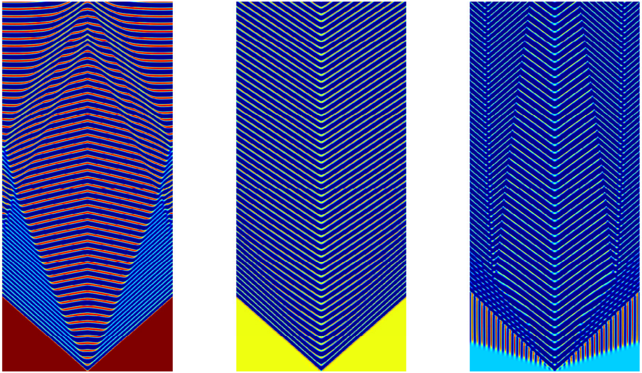


$a = 0.15$, $b = 0.05$, $c = 0.2$, $M = 5$ (left); $M = 10$ (middle); $M = 15$ (right)

Nonlocal reaction-diffusion model contd.
RM model with nonlocal interaction contd.

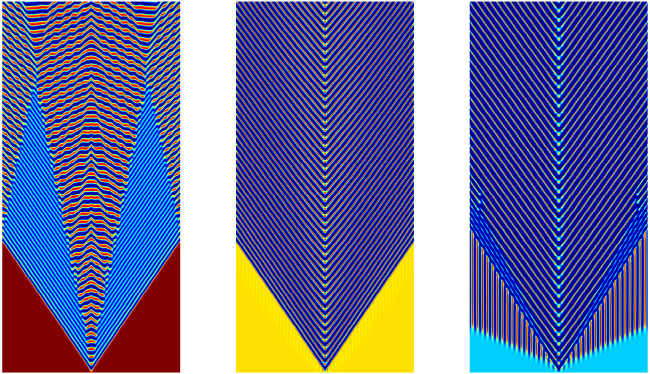


Nonlocal reaction-diffusion model contd.
RM model with nonlocal interaction contd.



$a = 0.9,\ b = 0.3,\ c = 0.2,\ M = 5$ (left); $M = 10$ (middle); $M = 15$ (right)

Nonlocal reaction-diffusion model contd.
RM model with nonlocal interaction contd.



$a = 1.0,\ b = 0.5,\ c = 0.2,\ M = 5$ (left); $M = 10$ (middle); $M = 15$ (right)

Nonlocal reaction-diffusion model contd.
RM model with nonlocal interaction contd.

Turing pattern in RM model

$$\frac{\partial n}{\partial t} = n\left(1 - \int_{-\infty}^{\infty} \phi(x-y)n(y,t)dy\right) - \frac{np}{c+n} + d\frac{\partial^2 n}{\partial x^2},$$

$$\frac{\partial p}{\partial t} = \frac{anp}{c+n} - bp + \frac{\partial^2 p}{\partial x^2},$$

Associated eigenvalue problem

$$\lambda n(x) = A_1 n(x) - A_2 p(x) - n_* \int_{-\infty}^{\infty} \phi(x-y)n(y)dy + dn''(x),$$

$$\lambda p(x) = B_1 n(x) + p''(x),$$

$$A_1 = \frac{n_* p_*}{(c+n_*)^2} > 0, \ A_2 = \frac{n_*}{c+n_*} > 0, \ B_1 = \frac{acp_*}{(c+n_*)^2} > 0.$$

Nonlocal reaction-diffusion model contd.
RM model with nonlocal interaction contd.

After taking Fourier transformation

$$\begin{aligned}\lambda\overline{n}(\xi) &= A_1\overline{n}(\xi) - A_2\overline{p}(\xi) - n_*\overline{\phi}(\xi)\overline{n}(\xi) - d\xi^2\overline{n}(\xi),\\ \lambda\overline{p}(\xi) &= B_1\overline{n}(\xi) - \xi^2\overline{p}(\xi),\end{aligned}$$

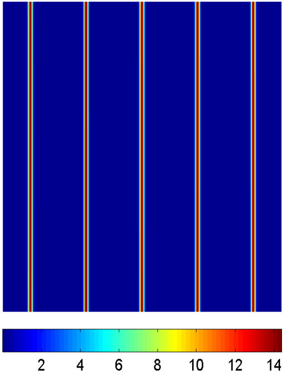
Characteristic equation

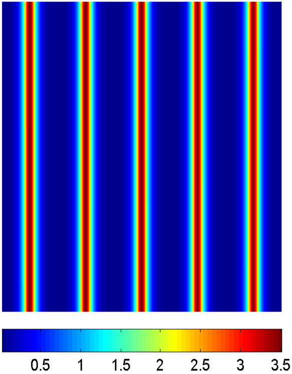
$$\lambda^2 + (n_*\overline{\phi} - A_1 + (d+1)\xi^2)\lambda + A_2B_1 + (n_*\overline{\phi} - A_1)\xi^2 + d\xi^4 = 0.$$

Conditions for stability of homogeneous steady-state

$$\begin{aligned}A_1 - n_*\overline{\phi} - (d+1)\xi^2 &< 0,\\ d\xi^4 + (n_*\overline{\phi} - A_1)\xi^2 + A_2B_1 &> 0.\end{aligned}$$

Nonlocal reaction-diffusion model contd.
RM model with nonlocal interaction contd.





$a = 1.2,\ b = 0.6,\ c = 0.4,\ d = 0.05$ and $M = 10$

Nonlocal reaction-diffusion model contd.
Bazykin model with nonlocal grazing

Bazykin model with nonlocal consumption and reproduction

$$\frac{\partial n(t,x)}{\partial t} = rn(t,x)\left(1 - \frac{n(t,x)}{k}\right) - \frac{bn(t,x)}{1+n(t,x)} w_1(t,x) + \frac{\partial^2 n(t,x)}{\partial x^2}$$

$$\frac{\partial p(t,x)}{\partial t} = bw_2(t,x)p(t,x) - p(t,x) - fp^2(t,x) + D_p \frac{\partial^2 p(t,x)}{\partial x^2}$$

$$n(0,x) = n_0(x),\ p(0,x) = p_0(x)$$

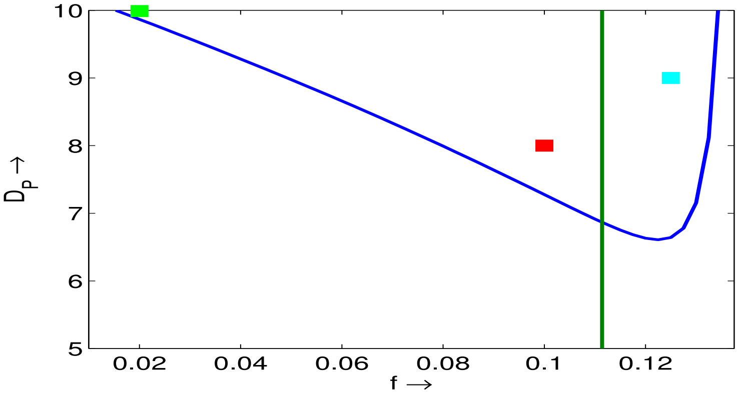
$$n(t,0) = n(t,L),\ p(t,0) = p(t,L)$$

$$w_1(t,x) = \int_{-\infty}^{\infty} \phi_1(x-y)p(t,y)dy$$

$$w_2(t,x) = \int_{-\infty}^{\infty} \phi_2(x-y)\frac{n(t,y)}{1+n(t,y)}dy$$

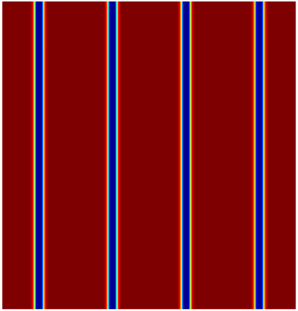
$$\phi_1(z) = \phi_2(z) = \phi_{step}(z)$$

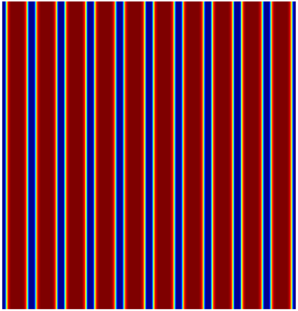
Nonlocal reaction-diffusion model contd.
Bazykin model with nonlocal grazing contd.



Bifurcation diagram for parameter values: $r = 0.24$, $k = 30$, $b = 1.3$ and $M = 0$

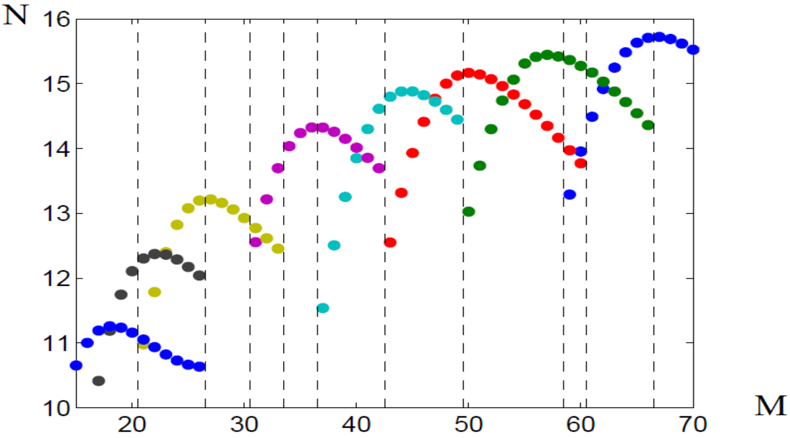
Nonlocal reaction-diffusion model contd.
Bazykin model with nonlocal grazing contd.





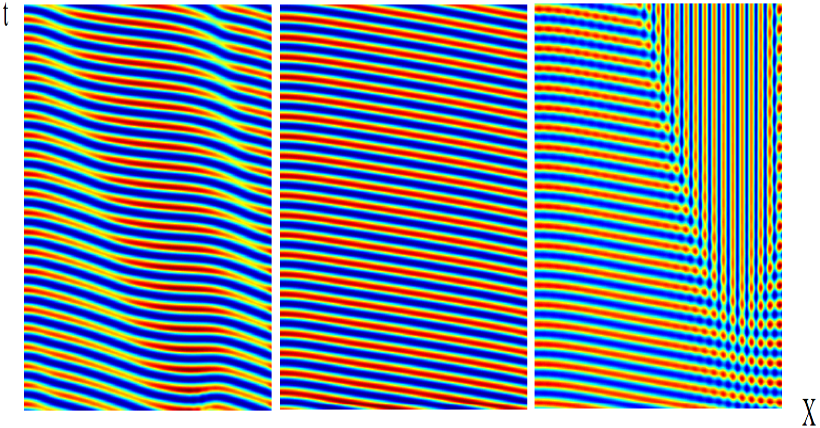
Parameter values: $r = 0.24$, $k = 30$, $b = 1.3$, $f = 1.25$, $D_p = 9$, $M = 20$, $M = 40$

Nonlocal reaction-diffusion model contd.
Bazykin model with nonlocal grazing contd.



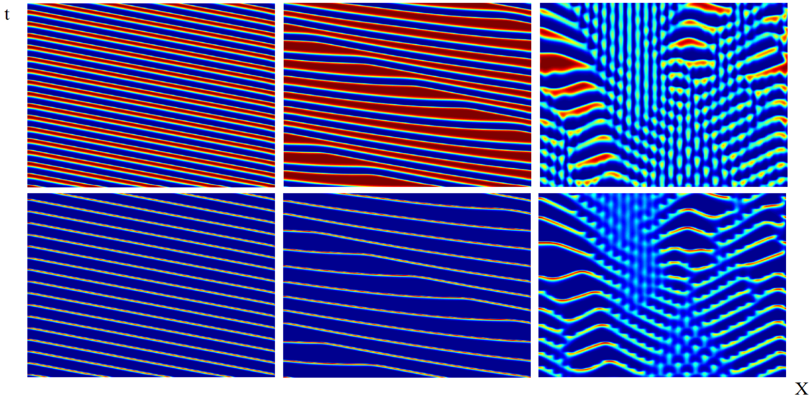
Existence of multiple stationaty steady-state

Nonlocal reaction-diffusion model contd.
Bazykin model with nonlocal grazing contd.



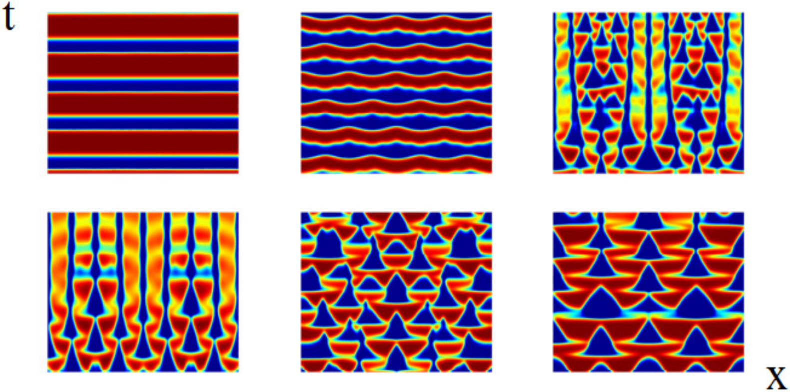
$r = 0.24,\ k = 30,\ b = 1.3,\ D_p = 8,\ M = 2,\ f = 0.098$ (left), $f = 0.1$ (middle), $f = 0.102$ (right)

Nonlocal reaction-diffusion model contd.
Bazykin model with nonlocal grazing contd.



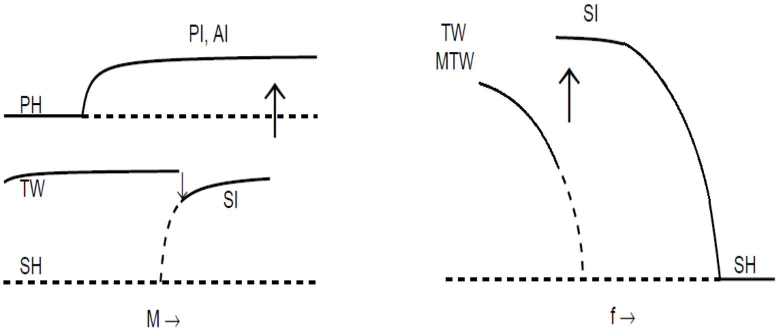
Travelling wave, modulated travelling waves, $r = 0.24$, $k = 30$, $b = 1.3$, $f = 0.02$, $D_p = 10$, $M = 2$ (left), $M = 5$ (middle), $M = 8$ (right)

Nonlocal reaction-diffusion model contd.
Bazykin model with nonlocal grazing contd.



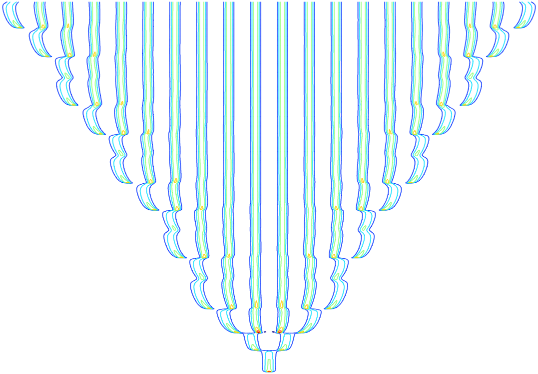
$r = 0.24$, $k = 30$, $b = 1.3$, $D_p = 10$, $f = 0.02$ and increasing values of M, from $M = 10$ to $M = 60$

Nonlocal reaction-diffusion model contd.
Bazykin model with nonlocal grazing contd.



SH - stationary homogeneous, SI - stationary inhomogeneous, PH - periodic homogeneous, PI - periodic inhomogeneous, AI - aperiodic inhomogeneous, TW - travelling waves, MTW - modulated travelling waves. Solid lines correspond to stable branches, dashed lines to unstable, arrows show transition between the branches

Nonlocal reaction-diffusion model contd.



Emergence of new patches OR Darwin's speciation (see V. Volpert: Partial Differential Equations, Vol - II)

Outline

Some comments & future goal

- Consideration of nonlocal interaction terms induces Turing patterns in RM model
- Consideration of nonlocal interaction can have a stabilizing effect, upto certain extent
- Linear stability analysis and related numerical simulation reveals the existence of multiple steady-states
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- Construction of global bifurcation diagram for the non-local spatio-temporal models
- This theory has wide applicability in system biology and various branches of natural sciences

Outline

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Thank You very much for tolerating me during last 45 mins.