Optimal Vaccination Strategies and Rational Behavior in Seasonal Epidemics

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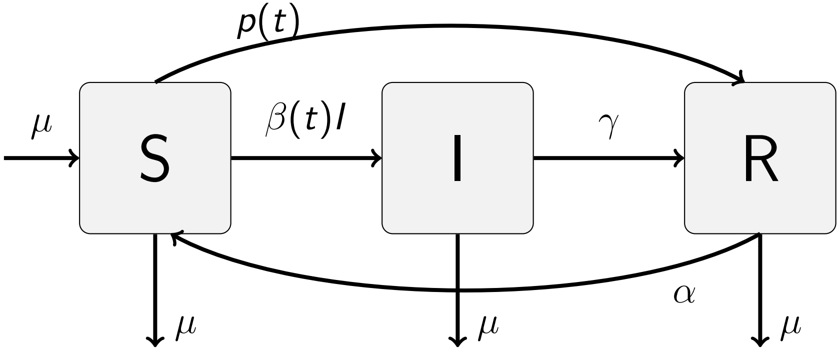
Vaccine wars
How it used to be

Vaccine Rebellion - Rio de Janeiro, 10-16 November 1904

Vaccine wars

Prevention and sceptiks

The model



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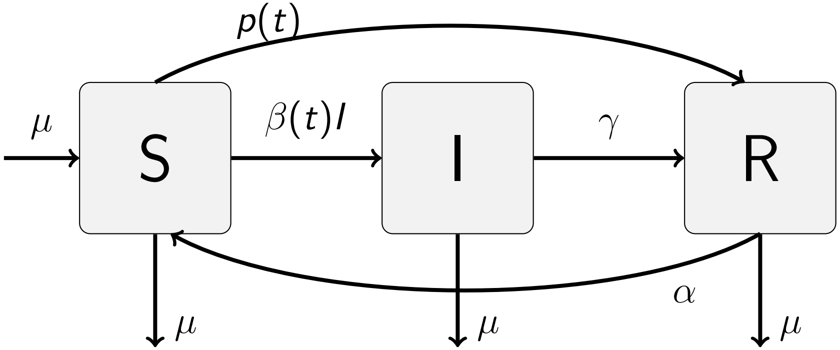

$$S' = \mu + \alpha R - \beta(t) IS - p(t)S - \mu S$$
$$I' = \beta(t) IS - \gamma I - \mu I$$
$$R' = \gamma I + p(t)S - \mu R - \alpha R$$

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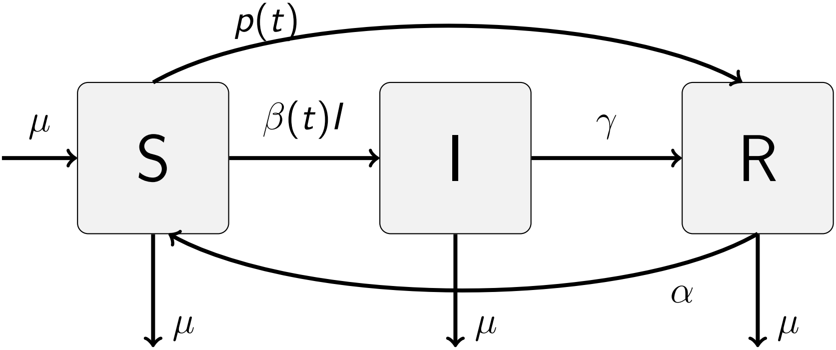

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$\beta(t+T) = \beta(t)$, $p(t+T) = p(t)$.
$\beta \in \mathrm{BV}([0,T])$, $p \in \mathrm{Radon}([0,T])$
$\int_0^T p\mathrm{d}t \leq (\alpha+\mu) T \sup_{[0,T]} \beta/(\gamma+\mu)$

Disease-free solution

Uniqueness and a lemma

Lemma

Assume $\beta(t+T) = \beta(t)$, $p(t+T) = p(t)$. Then, there is a unique disease-free solution $S_0(t)$ such that $S_0(t+T) = S_0(t)$. All initial conditions such that $I(0) = 0$ are attracted to this solution.

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Assume $\beta(t+T) = \beta(t)$, $p(t+T) = p(t)$*. Then, there is a unique disease-free solution* $S_0(t)$ *such that* $S_0(t+T) = S_0(t)$*. All initial conditions such that* $I(0) = 0$ *are attracted to this solution.*

Lemma

Consider that $\beta(t) = \beta_0 > 0$ *and* $p(t) = p_0 \geq 0$*. Then the three conditions below are equivalent:*

1. *The disease free solution* $(\hat{S}_0, 0)$ *is asymptotically stable.*
2. $\frac{\beta_0 \hat{S}_0}{\gamma+\mu} \leq 1$.
3. $I' < 0$ *for all* $I > 0$ *and all* $S < \hat{S}_0$.

Let χ_{p} be the set of preventive strategies, i.e.,

$$\chi_{\mathrm{p}} = \{p|\ I' < 0 \text{ for all } I > 0 \text{ and } S(0) < \min \hat{S}_0(t)\}.$$

We define the vaccination effort: $\mathbb{E}[p] = T^{-1} \int_0^T pS_0[p] \mathrm{d}t$.
We say that a strategy p_{opt} is optimal if

1. There is a sequence $p_n \in \chi_{\mathrm{p}}$ such that $p_n \to p_{\mathrm{opt}}$;
2. For any strategy $p \in \chi_{\mathrm{p}}$, $\mathbb{E}[p] \geq \mathbb{E}[p_{\mathrm{opt}}]$.

Theorem

For any β, there exists at least one optimal solution.

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Assume that $\beta'(t) \geq -(\mu+\alpha)\beta(t)\left(\frac{\beta(t)}{\gamma+\mu}-1\right)$. Then

$$p_{\rm opt}(t) = (\mu+\alpha)\left(\frac{\beta(t)}{\gamma+\mu}-1\right)+\frac{\beta'(t)}{\beta(t)}$$

is the optimal strategy.

Lemma

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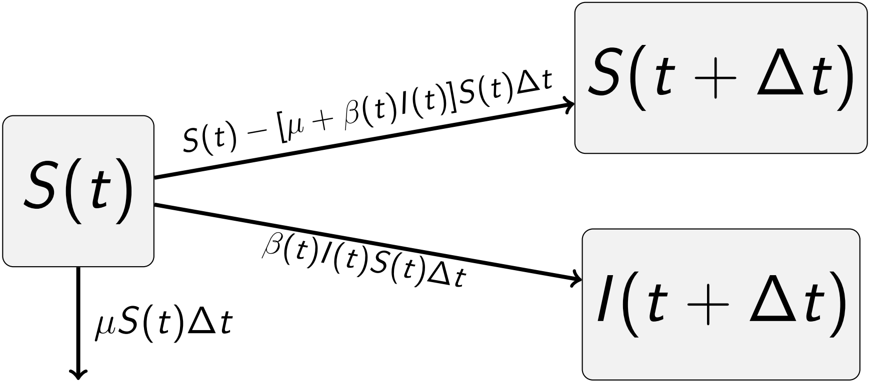

Let $r := r_{\mathrm{v}}/r_{\mathrm{d}}$ be the ratio between the risk of the vaccination and the risk of the disease.

p is a *herd immunity provider strategy* if $\beta(t)I_1[p](t) < r$ for all t. We call χ_{h} the set of herd immunity provider strategies.

A rational individual minimizes the joint risk of the disease and the vaccination: if $p \in \chi_{\mathrm{h}}$, a rational individual will not be vaccinated.

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$$\begin{aligned}\rho[p_*, p] &= r_{\mathrm{d}} \int_0^T \beta(t)I[p](t)\mathrm{e}^{-\int_0^t (p_*+\mu)\mathrm{d}t}\mathrm{d}t + r_{\mathrm{v}} \int_0^T \left(1 - \mathrm{e}^{-\int_0^t (p_*+\mu)\mathrm{d}t}\right)\mathrm{d}t \\ &= -r_{\mathrm{d}} \int_0^T \left(r - \beta(t)I[p](t)\right)\mathrm{e}^{-\int_0^t (p_*+\mu)\mathrm{d}t}\mathrm{d}t + r_{\mathrm{v}} T \ ,\end{aligned}$$

where p is the population strategy, while p_* is the strategy of the focal individual.

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where p is the population strategy, while p_* is the strategy of the focal individual. We say that p_Nash is a Nash-strategy if all individuals behave rationally.

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Assume

$$\frac{\beta'(t)}{\beta(t)} \leq \gamma + \mu \leq \frac{\beta'(t)}{\beta(t)} + \beta(t) \qquad \textit{and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{(r+\alpha-\gamma)t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{(\gamma+\mu)t}}{\beta(t)}\right)\right] \leq \mathrm{e}^{(r+\mu+\alpha)t}\left(\mu+\alpha-\frac{\alpha r}{\beta(t)}\right) .$$

Then, the strategy given by

$$p_{\mathrm{Nash}}(t) = \frac{\beta(t)^2}{(\gamma+\mu)\beta(t)-\beta'(t)}\left[\mu+\alpha+(\gamma+\mu)\frac{\beta'(t)}{\beta(t)^2} - 2\frac{\beta'(t)^2}{\beta(t)^3} + \frac{\beta''(t)}{\beta(t)^2} - \frac{\alpha r}{\beta(t)}\right] - [(1-\mu)r+\mu+\alpha]$$

is one Nash-equilibrium strategy.

Facts and comparisons

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4. Assume that $p_{\rm Nash} \notin \chi_{\rm p}$. Then, $\mathbb{E}[p_{\rm Nash}] < \mathbb{E}[p_{\rm opt}]$.
5. In the conditions of the explicit formulas:
   $\mathbb{E}[p_{\rm opt}] - \mathbb{E}[p_{\rm Nash}] = r(\gamma + \mu + \alpha)\langle\beta^{-1}\rangle > 0$.

Examples
The constant case

If $\beta < \gamma + \mu$, $p_{\rm opt} = 0$.
If $\beta > \gamma + \mu$,

$$p_{\rm opt} = \mu \left(\frac{\beta_0}{\gamma + \mu} - 1 \right)$$

$$p_{\rm Nash} = \mu \left(\frac{\beta_0}{\gamma + \mu} - 1 \right) - (1 - \mu) r = p_* - (1 - \mu) r < p_{\rm opt} \ .$$

Simulations

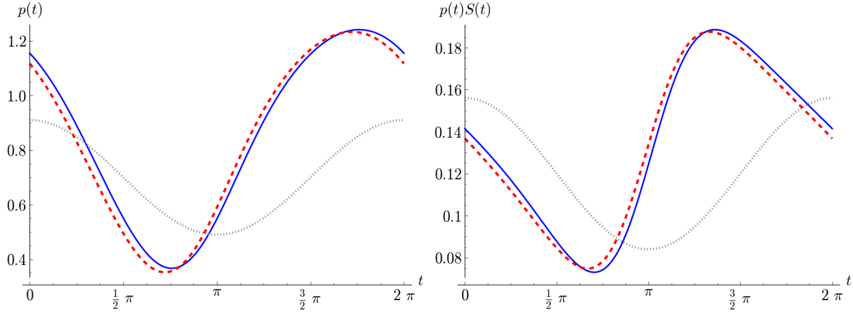

Figure : We use $\mu = (80T)^{-1}$, $\gamma = 52/T$, $\alpha = 1/T$, with $T = 2\pi$, $r = 0.01$ and $\beta(t) = 52(1 + 0.3\cos(t))$, implying $\frac{\gamma}{\beta_0\alpha} = 1$. Left: $p_{\mathrm{Nash}}(\text{red}) > p_{\mathrm{opt}}(\text{blue})$ in the beginning of the epidemic season and smaller otherwise. The peak p_{opt} is $\pi/4$ before the peak of the transmission rate; the peak of p_{Nash} is slightly before. Right: Time dependent vaccination effort, in the two cases. For simplicity, we plot in both cases $\beta(t)$ in dark dashed line (out of scale). The choice of parameters implies that, in the absence of vaccination, there is only one stable attractor.

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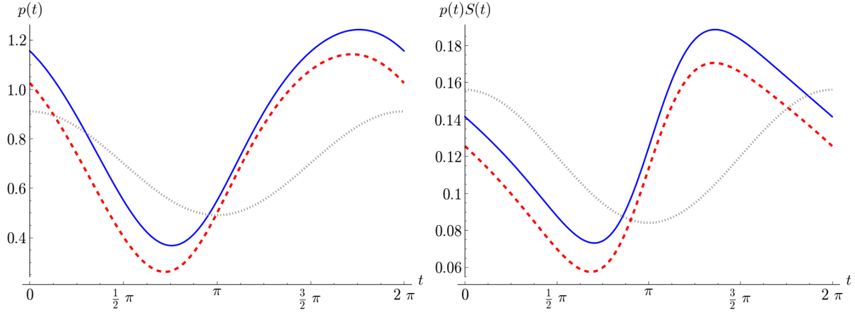

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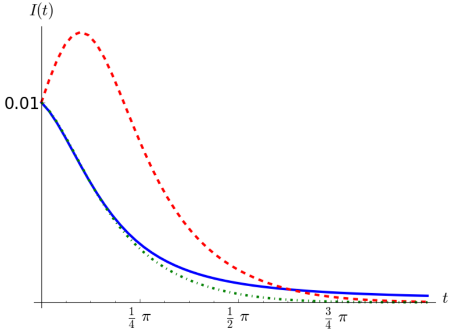

Figure : All parameters as before and assume initial conditions given by $I(0) = 0.01$. We consider three different vaccination profiles $p_{\mathrm{opt}}[\beta]$ (blue, continuous), $p_{\mathrm{opt}}[\langle\beta\rangle]$ (red, dashed) and $p_{\mathrm{opt}}[\sup\beta]$ (green, dash-dot). Note that the optimal strategy against the average transmission rate is unable to prevent initial outbreaks (however, $I(t) \to 0$ when $t \to \infty$). Vaccination efforts $\mathbb{E}[p_{\mathrm{opt}}[\sup\beta]] = 0.141 > \mathbb{E}[p_{\mathrm{opt}}[\langle\beta\rangle]] = 0.135 > \mathbb{E}[p_{\mathrm{opt}}[\beta]] = 0.133$, respectively.

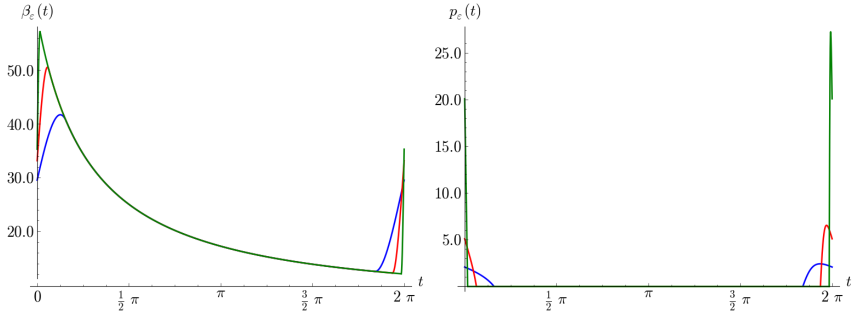

Figure : $\beta(t) = \frac{\gamma+\mu}{1-K\exp(-(\mu+\alpha)t))}$, $t \in [0, 2\pi)$, $K \in (0,1)$. We consider a sequence β_ε of differentiable functions, such that $\beta_\varepsilon(t) = \beta(t)$ for $t \in (\varepsilon, 2\pi - \varepsilon)$ and the points $2\pi - \varepsilon$ and $2\pi + \varepsilon \equiv \varepsilon$ are connected by a third order polinomial. Right: a sequence of $p_\varepsilon := p_{\rm opt}[\beta_\varepsilon]$; note that $(p_\varepsilon)_{\varepsilon>0}$ ressembles a delta-sequence.

Conclusions

... and future work

1. We considered two extreme definitions of vaccination strategies.

Further informations:

http://arxiv.org/abs/1507.02940

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