The existence of multiple decisions for vaccination in the reinfection SIRI model

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1. Introduction

- In the case of voluntary vaccination, individuals take into account different aspects to decide between vaccinate or not:
  - the probability of become infected;
  - the adverse consequences that might result from such infection and also from vaccination, i.e. **the morbidity risks**.
- The decision of each individual is also influenced by the decisions of all other individuals.
- Bauch and Earn (2004) used the SIR model to do a game theoretical approach to study the impact of the changes of the morbidity relative risk on the individual's decisions.
- Here, we consider the $SIRI$ model that incorporates in the SIR model the effects of reinfection due to partial immunity.
- The presence of partial immunity, introduces the co-existence of two scenarios with relevant and opposite features for the same level of risk: the **low-vaccination** and the **high-vaccination** scenarios.

2. Vaccination Nash and ESV strategies

- For simplicity, we assume that all individuals are provided with the same information and use this information in the same way to assess risks.
- An individual's strategy is the probability P that s/he will choose to vaccinate.
- The population vaccination strategy p is the proportion of individuals who will be vaccinated and hence is the mean of all strategies adopted by the individuals in the population.

Let:

- r_v be the morbidity risks from vaccination , i.e the probability of adverse consequences to vaccination;
- r_i be the morbidity risks from infection , i.e the probability of adverse consequences to infection;
- $r = r_v/r_i$ be the *morbidity relative risk*;

- $\pi^p_{\bar{v}}$ be the probability that an non vaccinated individual will eventually be infected if the vaccine coverage level in the population is p;
- π^p_v be the probability that a vaccinated individual will eventually be infected if the vaccine coverage level in the population is p;
- $\pi(p) = \pi^p_{\bar{v}} - \pi^p_v$ be the *vaccination infection risk index*.

Hence, with this notation we define:

- the payoff to an non-vaccinated individual: $-r_i\pi^p_{\overline{v}}$
- the payoff to a vaccinated individual: $-r_v - r_i\pi^p_v$

The *vaccination expected payoff* $E(P,p) \equiv E(P,p;r)$ is,

$$\begin{aligned} E(P,p) &= \frac{(-r_v - r_i\pi_v(p))P + (-r_i\pi_{\overline{v}}(p))(1-P)}{r_i} \\ &= -(r + \pi_v(p))P - \pi_{\overline{v}}(p)(1-P) \\ &= -\pi_{\overline{v}}(p) + (\pi(p) - r)P \, . \end{aligned}$$

Using the usual concepts of game theory, we will define the Nash and the evolutionary stable vaccination strategies that are more likely to be adopted by the individuals.

Definition

For a given relative morbidity risk $r \geq 0$, the population vaccination strategy P^* is a *vaccination Nash equilibrium*, if

$$E(Q, P^*) - E(P^*, P^*) = (\pi(P^*) - r)(Q - P^*) \leq 0 \,, \qquad (1)$$

for every strategies $Q \in [0, 1]$.

- Hence, if the population vaccination strategy is the Nash equilibrium P^* then no single individual has the incentive to change its strategy from P^*.

Lemma (Nash equilibria)

Let us assume that the vaccination-infection risk index π is continuous. The population vaccination strategy P^* is a Nash equilibrium if, and only if, P^* satisfies one of the following conditions:

(i) $P^* = 0$ and $r \geq \pi(0)$; or

(ii) $P^* \in (0, 1)$ and $r = \pi(P^*)$; or

(iii) $P^* = 1$ and $r \leq \pi(1)$.

Now, suppose that all individuals were opting by an individual vaccination strategy P and consider that a group, of size ε, opt for an individual vaccination strategy Q.
The new vaccination population strategy is

$$p(\varepsilon) = (1-\varepsilon)P + \varepsilon Q = P + \varepsilon(Q - P).$$

The vaccination expected payoff of the individuals with vaccination strategy P is

$$E(P, p(\varepsilon)) = -\pi_{\overline{v}}(p(\varepsilon)) + (\pi(p(\varepsilon)) - r)P\,;$$

and with vaccination strategy Q is

$$E(Q, p(\varepsilon)) = -\pi_{\overline{v}}(p(\varepsilon)) + (\pi(p(\varepsilon)) - r)Q.$$

We observe that both vaccination expected payoffs depend upon the vaccination strategy of the individuals, P and Q, and on the sizes of the groups, $1-\varepsilon$ and ε.

The *vaccination expected payoff gain* function $\Delta E_{P\to Q}(p(\varepsilon))$ of moving from the vaccination strategy P to Q is

$$\Delta E_{P\to Q}(p(\varepsilon)) = E(Q, p(\varepsilon)) - E(P, p(\varepsilon)) = (\pi(p(\varepsilon)) - r)\,(Q - P)\,.$$

- $\Delta E_{P\to Q}(p(\varepsilon))$ measures the incentive that a group, of size ε, has to change his vaccination strategy from P to Q.

Definition

For a given relative morbidity risk $r \geq 0$, the population vaccination strategy P^* is an *evolutionary stable vaccination (ESV)* strategy, if there is a $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0)$ and for every $Q \in [0,1]$, with $Q \neq P^*$,

$$\Delta E_{P^*\to Q}(p(\varepsilon)) < 0\,.$$

- Hence, the population vaccination strategy P^* is an ESV strategy if any small group of individuals that try to adopt a different strategy Q obtain a lower payoff than those adopting P^*.

Lemma (ESV strategies)

Let us assume that the vaccination-infection risk index π is continuous. A population vaccination strategy P^* is an ESV strategy if, and only if, P^* satisfies one of the following conditions:

(i) $P^* = 0$ and $r > \pi(0)$; or

(ii) $P^* \in [0, 1]$, $r = \pi(P^*)$ and π is strictly decreasing at P^*; or

(iii) $P^* = 1$ and $r < \pi(1)$.

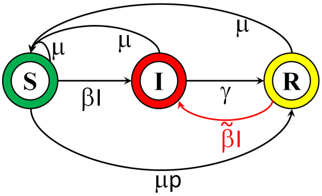
Furthermore, a strategy P^* is a Nash equilibrium that is not an ESV strategy if, and only if, P^* satisfies the following condition:

(iv) $P^* \in [0, 1]$, $r = \pi(P^*)$ and π is not strictly decreasing at P^*.

3. Vaccination expected payoff for the SIRI model

The SIRI epidemiological model is described by the ODE system:

$$\frac{dS}{dt} = \mu(1-p) - \beta SI - \mu S$$
$$\frac{dI}{dt} = \beta SI - (\mu+\gamma)I + \tilde{\beta}RI$$
$$\frac{dR}{dt} = \mu p + \gamma I - \tilde{\beta}RI - \mu R$$



where,

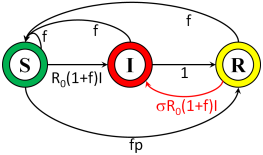
- μ is the mean birth and death rate
- β is the mean infection rate
- $\tilde{\beta}$ is the mean reinfection rate
- $1/\gamma$ is the mean infectious period
- p is the vaccine uptake level

- S are the non-vaccinated individuals
- R are the vaccinated individuals

Since $S + I + R = 1$, the remaining two equations can be written in a convenient dimensionless form

$$\frac{dS}{d\tau} = f(1-p) - (\tilde{R}_0 SI + fS)$$

$$\frac{dI}{d\tau} = \tilde{R}_0 SI + \sigma \tilde{R}_0 RI - (1+f)I$$



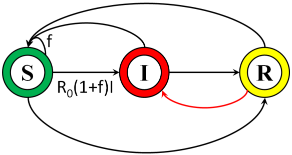
where,

- $\tau = t/\gamma$ is time measured in units of the mean infectious period
- $f = \mu/\gamma$ is the infectious period as a fraction of mean lifetime
- $R_0 = \beta/(\gamma+\mu)$ is the **basic reproductive ratio** - the average number of secondary cases produced by a typical primary case in a fully susceptible population
- $\tilde{R}_0 = (1+f)R_0$ is the adapted basic reproductive number
- $\sigma = \tilde{\beta}/\beta$ **is the ratio between infection and reinfection rates.**

Now, we use the stationary states S^*, I^* and R^* to obtain $\pi^p_{\overline{v}}$ and π^p_v:

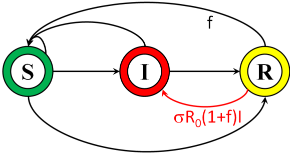
$\pi_{\overline{v}}(p)$ is the ratio between the susceptible individuals that become infected $-\tilde{R}_0 SI$ and all the individuals that leave the susceptible class without vaccination $-(\tilde{R}_0 SI + fS)$, i.e.

$$\pi_{\overline{v}}(p) = \frac{\tilde{R}_0 SI}{\tilde{R}_0 SI + fS} = \frac{\tilde{R}_0 I}{\tilde{R}_0 I + f} \, .$$



and $\pi_v(p)$ is

$$\pi_v(p) = \frac{\sigma \tilde{R}_0 RI}{\sigma \tilde{R}_0 RI + fR} = \frac{\sigma \tilde{R}_0 I}{\sigma \tilde{R}_0 I + f} \, .$$



Hence, for the SIRI model, the vaccination-infection risk index is

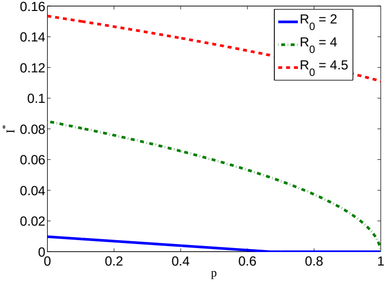
$$\pi(p) = \pi_{\overline{v}}(p) - \pi_v(p) = \frac{\tilde{R}_0 I}{\tilde{R}_0 I + f} - \frac{\sigma \tilde{R}_0 I}{\sigma \tilde{R}_0 I + f} = \frac{f}{\sigma \tilde{R}_0 I + f} - \frac{f}{\tilde{R}_0 I + f},$$

and the vaccination expected payoff $E(P,p) \equiv E(P,p;r,R_0)$ is

$$E(P,p) = -\frac{\tilde{R}_0 I}{\tilde{R}_0 I + f} + \left(\frac{f}{\sigma \tilde{R}_0 I + f} - \frac{f}{\tilde{R}_0 I + f} - r \right) P \,.$$

The *critical vaccine uptake level* $p_c : (0, +\infty) \to [0,1] \cup \{+\infty\}$ is

$$p_c(R_0) = \begin{cases} 0 & \text{if} \quad R_0 \leq 1 \\ \frac{R_0 - 1}{R_0(1-\sigma)} & \text{if} \quad 1 < R_0 \leq \frac{1}{\sigma} \\ +\infty & \text{if} \quad R_0 > \frac{1}{\sigma} \end{cases} .$$



$I^*(p)$, *for* $f = 0.01$, $\sigma = 0.25$.

$p_{crit}(R_0 = 2) = 2/3$, $p_{crit}(R_0 = 4) = 1$ *and* $\nexists\, p_{crit}(R_0 = 4.5)$.

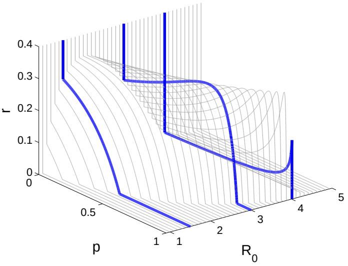
- For $R_0 > 1/\sigma$, even if $p = 1$ the disease is not eliminated!

The value $R_0 = 1/\sigma$ is the **reinfection threshold**.

4. Vaccination scenarios

Now, we will study the Nash and the ESV strategies effects in the vaccination population strategy depending upon the perceived relative risk r and upon the basic reproductive number R_0.

- First, we study the ideal vaccination scenario free of perceived morbidity risks $r = 0$.
- Secondly, we study the effect of the perceived morbidity risks $r > 0$ in the Nash and ESV strategies of the population.



The vaccination Nash equilibrium strategies for several R_0, with $R_0 = 1.5$, $R_0 = 3$ and $R_0 = 4.01$ highlighted.

Other parameters: $f = 0.001$ and $\sigma = 0.25$.

Theorem (Free of relative morbidity risks)

Suppose that $r = 0$. The Nash equilibria are the following:
For $R_0 < 1/\sigma$:

(i) *p_c is a left EVS strategy and a right weak EVS strategy; and*

(ii) *$p^* \in (p_c, 1]$ are weak EVS strategies.*

For $R_0 \geq 1/\sigma$:

(iii) *$P^* = 1$ is an ESV strategy.*

For positive morbidity relative risks $r > 0$, we will consider small and large reproductive ratios separated by

$$1 < R_B = \frac{1}{\sqrt{\sigma}} + \frac{f}{1+f} < \frac{1}{\sigma}$$

- for small reproductive ratios $R_0 \leq R_B$, there a single ESV strategy;
- for large reproductive ratios $R_0 > R_B$, there is a low and a high ESV strategy.

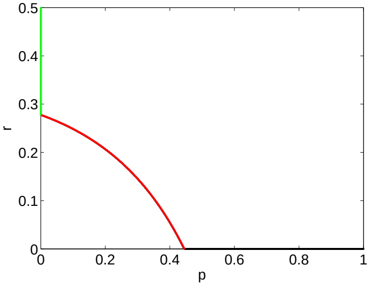
Theorem

For small basic reproductive ratios $R_0 \in (1, R_B]$ and positive morbidity relative risks $r > 0$, the Nash equilibria are

(i) $P^* = 0$ if $r \geq \pi(0)$

(ii) $P^* = \pi^{-1}(r)$ if $0 < r < \pi(0)$.

Furthermore, P^* are ESV strategies.



For small values of R_0 (here $R_0 = 1.5$), there is a

- **unique ESVS**

for each positive morbidity relative risk $r > 0$.

Now, let p_M be the vaccination strategy where the vaccination-infection risk index π attains its maximum

$$\frac{\partial \pi}{\partial p}(p_M) = 0.$$

We observe that $p_M \in (0,1)$ if

$$R_B < R_0 < R_C,$$

with

$$R_B = \frac{1}{\sqrt{\sigma}} + \frac{f}{1+f} \quad \text{and} \quad R_C = \frac{1}{\sigma} + \frac{f}{1+f}\frac{1}{\sqrt{\sigma}}\,.$$

Remark: $1 < R_B < 1/\sigma < R_C$.

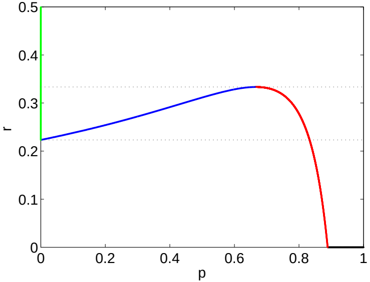
Theorem

For the large basic reproductive ratios $R_0 \in (R_B, R_C)$ and positive morbidity relative risks $r > 0$, the Nash equilibria are the following:

(i) the low ESVS $P^* = 0$ if $r > \pi(0)$;

(ii) the high ESVS $P^* = \pi^{-1}(r) \in (p_M, p_c]$ if $r < \pi(P_M)$;

(iii) the Nash equilibrium, that are not ESVS, $P^* = \pi^{-1}(r) \in [0, p_M]$ if $\pi(0) \leq r \leq \pi(P_M)$.



For large values of R_0 (here $R_0 = 3$) and some morbidity relative risks $r > 0$, we observe the co-existence of a

- **low-vaccination scenario**
- **high-vaccination scenario**

with opposite features.

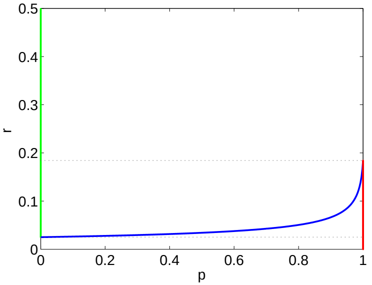
Theorem

For the large basic reproductive ratios $R_0 \geq R_C$ and positive morbidity relative risks $r > 0$, the Nash equilibria are the following:

(i) the low ESVS $P^* = 0$ if $r > \pi(0)$;

(ii) the high ESVS $P^* = 1$ if $r < \pi(1)$;

(iii) the Nash equilibrium, that are not ESVS, $P^* = \pi^{-1}(r) \in [0, 1]$ if $\pi(0) \leq r \leq \pi(1)$.



For large values of R_0 (here $R_0 = 4.01$) and some morbidity relative risk $r > 0$, we observe the co-existence of the

- **low-vaccination scenario** $P^* = 0$
- **high-vaccination scenario** $P^* = 1$.

5. Vaccination dynamics

We consider that a small group, of size ε, opts to change its vaccination strategy from the population vaccination strategy P to $P + \Delta P$.
The payoff gain function satisfies

$$\frac{\Delta E_{P\to(P+\Delta P)}}{\Delta P} = E(P+\Delta P, p) - E(P, p) = \pi(p) - r \ ,$$

where

$$p = p(\varepsilon) = (1-\varepsilon)P + \varepsilon(P + \Delta P).$$

Hence, we define the vaccination dynamics by

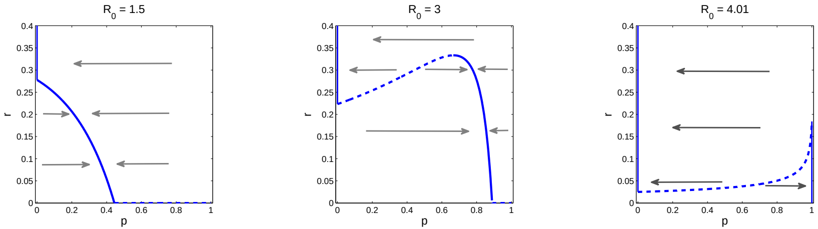
$$\frac{dp}{d\tau} = \alpha(p) \lim_{\Delta P\to 0} \frac{\Delta E_{P\to(P+\Delta P)}}{\Delta P} = \alpha(p)(\pi(p) - r) \ , \qquad (2)$$

where $\alpha(p) \geq 0$ measures the *vaccination strategy adaptation speed* of the population.

For the vaccination dynamics, we observe that:

i. the ESV strategies are attractors of the dynamics;

ii. the Nash equilibria that are not ESV strategies are boundaries of the basin of attractions of the ESV strategies.



The stable (solid line) and unstable (dashed line) equilibria of vaccination dynamics for $R_0 = 1.5$, $R_0 = 3$ and $R_0 = 4.01$.

References

- Chris T. Bauch and David J. D. Earn, Vaccination and the theory of games, PNAS 101 (2004) 13391–13394.
- J. Martins, A. Pinto (2015) *Co-existence of opposite evolutionary stable vaccination strategies in the reinfection SIRI model*, submitted.