Escola de Ciências e Tecnologia
Universidade de Évora

Polymatrix Games and Replicators

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DSABNS

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- Example

Lotka-Volterra System

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$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n\}$ is invariant under (1)

Replicator Equation

$$x_i' = x_i \left((Ax)_i - x^T A\,x\right), \quad i = 1, \ldots, n \tag{2}$$

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$\Delta^{n-1} = \{x \in \mathbb{R}^n : x_i \ge 0\,,\ \sum_{i=1}^n x_i = 1\}$ is invariant under (2)

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J. Hofbauer (1981): every LV system in $\mathbb{R}^n_+$ is orbit equivalent to a replicator system on the n-dimensional simplex Δ^n

Bimatrix replicator

$$\begin{cases} x_i' &= x_i\,\left((Ay)_i - x^t\,A\,y\right) \quad i = 1,\ldots,n \\ y_j' &= y_j\,\left((Bx)_j - y^t\,B\,x\right) \quad j = 1,\ldots,m \end{cases} \tag{3}$$

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$$\begin{array}{cc} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \left[\begin{array}{c|c} 0 & A_{n\times m} \\ \hline B_{m\times n} & 0 \end{array}\right]_{(n+m)\times(n+m)} \end{array}$$

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A polymatrix game is an ordered pair $(\underline{n}, A)$ where $\underline{n} = (n_1, \ldots, n_p)$ is a list of positive integers, called the game type, and $A \in \mathcal{M}_n(\mathbb{R})$ a square matrix of dimension $n = n_1 + \ldots + n_p$.

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$$
\begin{array}{c} \\ n_1 \\ \vdots \\ n_\alpha \\ \vdots \\ n_p \end{array}
\begin{array}{c}
\begin{array}{ccccc} n_1 & \ldots & n_\beta & \ldots & n_p \end{array} \\
\left[\begin{array}{c|c|c|c|c}
A^{1,1} & \ldots & A^{1,\beta} & \ldots & A^{1,p} \\ \hline
\vdots & \ddots & \vdots & \ddots & \vdots \\ \hline
A^{\alpha,1} & \ldots & A^{\alpha,\beta} & \ldots & A^{\alpha,p} \\ \hline
\vdots & \ddots & \vdots & \ddots & \vdots \\ \hline
A^{p,1} & \ldots & A^{p,\beta} & \ldots & A^{p,p}
\end{array}\right]_{n \times n}
\end{array}
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$\sum_{j \in \alpha} x_j\,(A\,x)_j$ represents the average payoff of all strategies in the group α

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$(A^{\alpha,\alpha} = 0,\ \alpha = 1,\ldots,p)$ replicator eq. for n-person games

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$\Gamma_{\underline{n}} := \Delta^{n_1-1} \times \ldots \times \Delta^{n_p-1} \subset \mathbb{R}^n$

$\Gamma_{\underline{n}}$ is parallel to $\quad H_{\underline{n}} := \left\{ x \in \mathbb{R}^n :\ \sum_{j\in\alpha} x_j = 0,\ \ \text{for}\ \ \alpha = 1,\ldots,p \right\}$

Polymatrix Replicator - Interior equilibria

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Proposition

Given a polymatrix game $(\underline{n}, A)$, a point $q \in \mathrm{int}(\Gamma_{\underline{n}})$ is an equilibrium of $X_{\underline{n},A}$ if and only if $(A\,q)_i = (A\,q)_j$ for all $i, j \in \alpha$ and $\alpha = 1, \ldots, p$.

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We call diagonal matrix of type $\underline{n}$ to any diagonal matrix $D = \mathrm{diag}(d_i)$ s.t. $d_i = d_j$ for all $i, j \in \alpha$ and $\alpha = 1, \ldots, p$.

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$$\begin{array}{c|c|c|c|c|c|}
 & n_1 & \dots & n_\alpha & \dots & n_p \\
\hline
n_1 & d_1 I_{n_1} & \dots & 0 & \dots & 0 \\
\hline
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\hline
n_\alpha & 0 & \dots & d_\alpha I_{n_\alpha} & \dots & 0 \\
\hline
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\hline
n_p & 0 & \dots & 0 & \dots & d_p I_{n_p}
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Conservative Polymatrix Replicators

Quadratic form $Q_A : H_{\underline{n}} \to \mathbb{R}\,,\ \ Q_A(w) := w^T A\, w$

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A polymatrix game $(\underline{n}, A)$ is called conservative if it has a formal equilibrium q, and there exists a positive diagonal matrix D of type $\underline{n}$ s.t. $Q_{A\,D} = 0$ on $H_{\underline{n}}$.

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Proposition

If $(\underline{n}, A)$ is conservative, q a formal equilibrium, and D a p.d.m. of type $\underline{n}$ s.t. $Q_{A\,D} = 0$ on $H_{\underline{n}}$, then

$$h(x) = -\sum_{i=1}^{n} \frac{q_i}{d_i} \log x_i$$

is a first integral for the flow of $X_{\underline{n},A}$, i.e., $\dot{h} = 0$ along the flow of $X_{\underline{n},A}$.

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is a first integral for the flow of $X_{\underline{n},A}$, i.e., $\dot{h} = 0$ along the flow of $X_{\underline{n},A}$. Moreover, $X_{\underline{n},A}$ is Hamiltonian w.r.t. a stratified Poisson structure on the prism $\Gamma_{\underline{n}}$, having h as its Hamiltonian function.

Dissipative Polymatrix Replicators

Definition

A polymatrix game $(\underline{n}, A)$ is called dissipative if it has a formal equilibrium q, and there exists a positive diagonal matrix D of type $\underline{n}$ s.t. $Q_{A\,D} \leq 0$ on $H_{\underline{n}}$.

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Proposition (Lyapunov function)

If $(\underline{n}, A)$ is dissipative, q a formal equilibrium and D a p.d.m. of type $\underline{n}$ s.t. $Q_{A\,D} \leq 0$ on $H_{\underline{n}}$, then

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is a Lyapunov function for the flow of $X_{\underline{n},A}$.

Dissipative Polymatrix Replicators

Proposition (Invariant Foliation)

Given a dissipative polymatrix game $(\underline{n}, A)$, if $X_{\underline{n},A}$ admits a formal equilibrium q, then there exists a $X_{\underline{n},A}$-invariant foliation $\mathscr{F}$ on $\mathrm{int}(\Gamma_{\underline{n}})$.

Dissipative Polymatrix Replicators

Proposition (Invariant Foliation)

Given a dissipative polymatrix game $(\underline{n}, A)$, if $X_{\underline{n},A}$ admits a formal equilibrium q, then there exists a $X_{\underline{n},A}$-invariant foliation $\mathscr{F}$ on $\mathrm{int}(\Gamma_{\underline{n}})$. Moreover, if $q \in \mathrm{int}(\Gamma_{\underline{n}})$, then every leaf of $\mathscr{F}$ contains exactly one equilibrium point.

Polymatrix Game - Example

Consider a population divided in 3 groups

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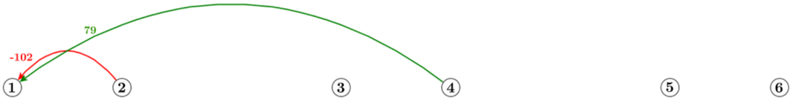

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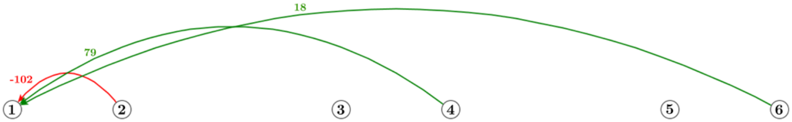

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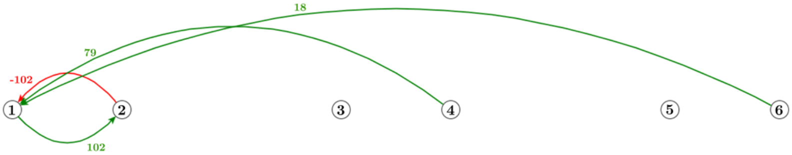

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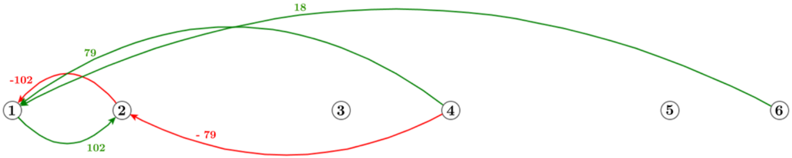

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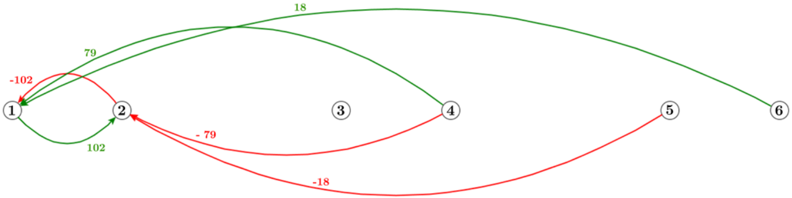

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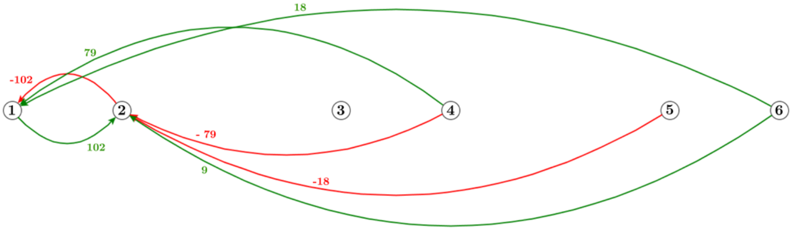

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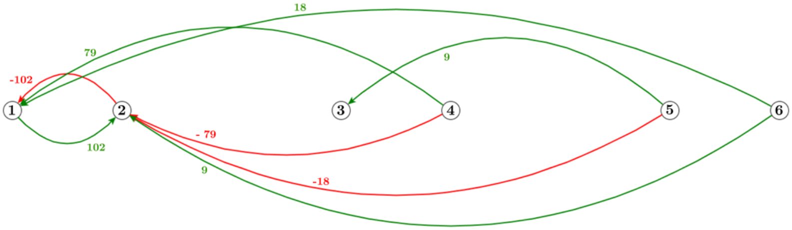

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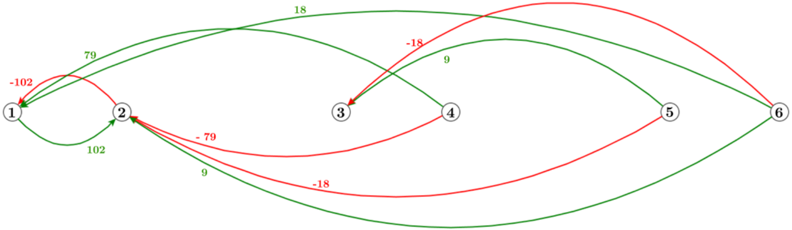

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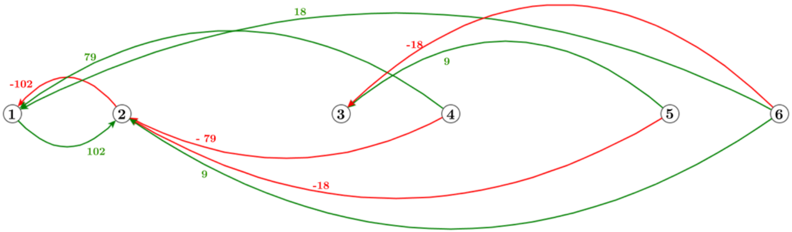

Polymatirx game $\mathcal{G} = ((2, 2, 2), A)$

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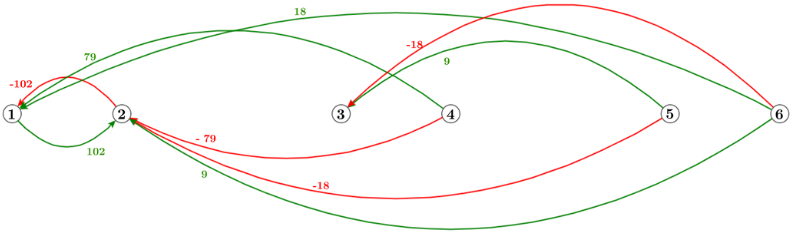

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$$A = \left[\begin{array}{cc|cc|cc} 0 & -102 & 0 & 79 & 0 & 18 \\ 102 & 0 & 0 & -79 & -18 & 9 \\ \hline 0 & 0 & 0 & 0 & 9 & -18 \\ -51 & 51 & 0 & 0 & 0 & 0 \\ \hline 0 & 102 & -79 & 0 & -18 & -9 \\ -102 & -51 & 158 & 0 & 9 & 0 \end{array}\right].$$

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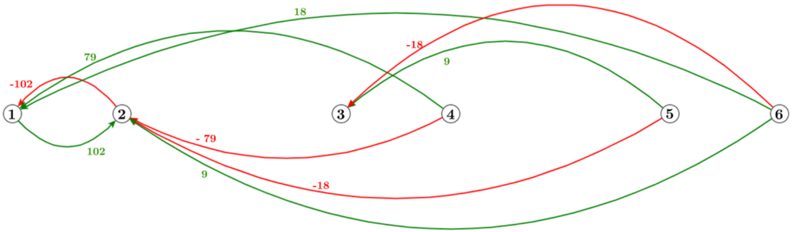

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$$A = \left[\begin{array}{cc|cc|cc} 0 & -102 & 0 & 79 & 0 & 18 \\ 102 & 0 & 0 & -79 & -18 & 9 \\ \hline 0 & 0 & 0 & 0 & 9 & -18 \\ -51 & 51 & 0 & 0 & 0 & 0 \\ \hline 0 & 102 & -79 & 0 & -18 & -9 \\ -102 & -51 & 158 & 0 & 9 & 0 \end{array}\right].$$

$X_{\mathcal{G}}$ vector field associated to the polymatrix replicator

Polymatrix Game - Example State Space

$$\Gamma_{(2,2,2)} = \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^3$$

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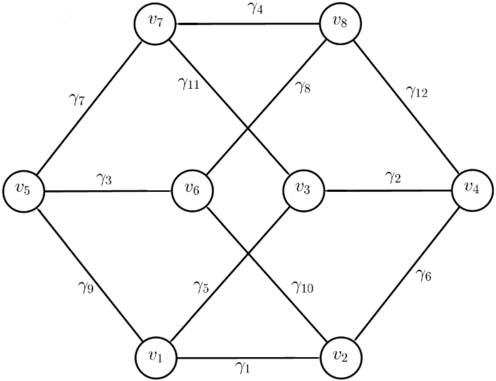

Figure: State space of $\mathcal{G}$.

Polymatrix Game - Example Equilibria

The associated polymatrix replicator has one interior equilibrium

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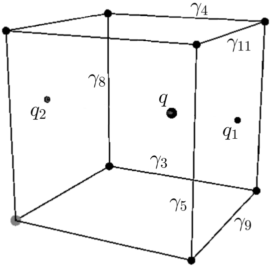

Figure: The equilibria of the associated polymatrix replicator of $\mathcal{G}$.

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where

$$D = \left[\begin{array}{cc|cc|cc} \frac{1}{51} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{51} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{79} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{79} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \end{array}\right]$$

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By Proposition (Lyapunov Function), this system admits a strict global Lyapunov function

$$h : \mathrm{int}\left(\Gamma_{(2,2,2)}\right) \to \mathbb{R}$$

for $X_{\mathcal{G}}$.

Polymatrix Game - Example Dynamics

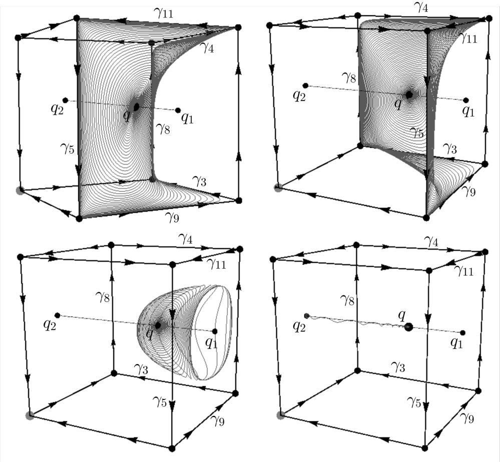

Figure: An approximation of the $X_{\mathcal{G}}$-invariant manifold from two different perspectives (up), and two different orbits starting near the respective faces equilibrium (down).

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