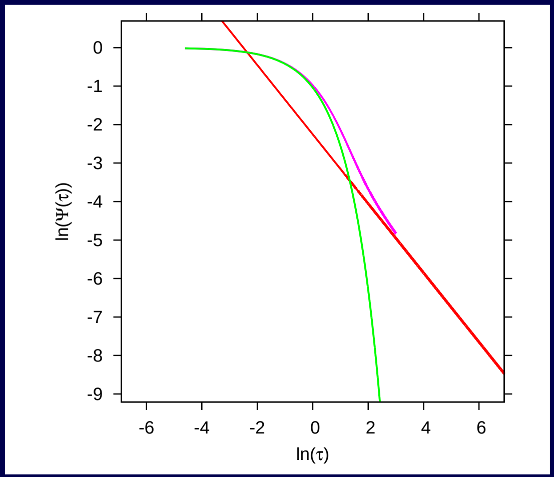
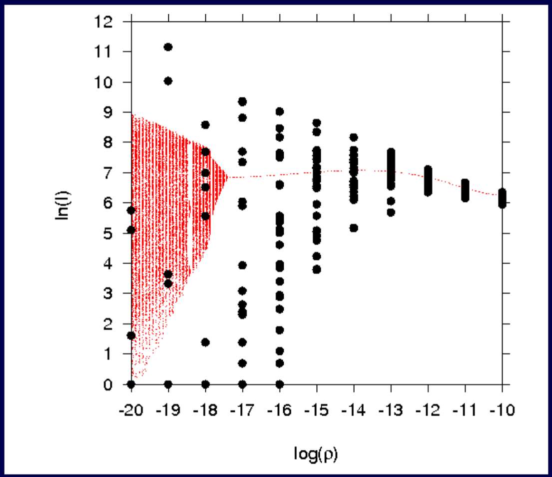
Power law jumps and power law waiting times, fractional calculus and human mobility in epidemiological systems





Nico Stollenwerk

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Univ. Lisboa

Simplest epidemiological processes

epidemic process: SIS

$$\begin{aligned} S + I &\xrightarrow{\beta} I + I \\ I &\xrightarrow{\alpha} S \end{aligned}$$

epidemic process: SIR

$$\begin{aligned} S + I &\xrightarrow{\beta} I + I \\ I &\xrightarrow{\gamma} R \\ R &\xrightarrow{\alpha} S \end{aligned}$$

Susceptible-Infected-Recovered epidem.: SIR

$$S + I \xrightarrow{\beta} I + I$$
$$I \xrightarrow{\gamma} R$$
$$R \xrightarrow{\alpha} S$$

gives ODE system

$$\frac{d}{dt} S = \alpha R - \frac{\beta}{N} S I$$
$$\frac{d}{dt} I = \frac{\beta}{N} S I - \gamma I$$
$$\frac{d}{dt} R = \gamma I - \alpha R$$

SIR system with seasonality

$$\frac{d}{dt}\,S = \alpha R - \frac{\beta(t)}{N}\,SI$$

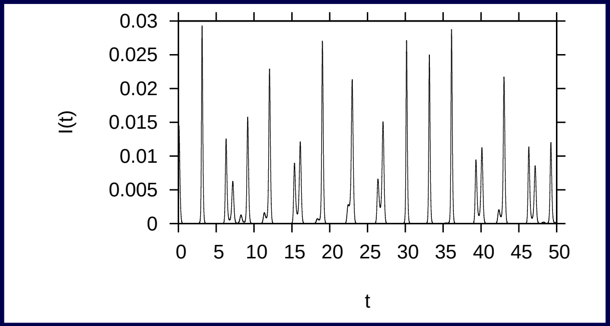
$$\frac{d}{dt}\,I = \frac{\beta(t)}{N}\,SI - \gamma I$$

$$\frac{d}{dt}\,R = \gamma I - \alpha R$$

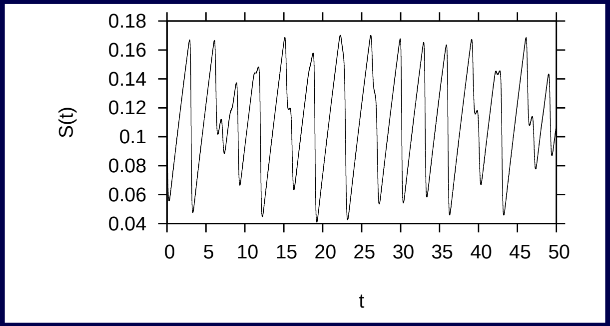
with seasonal forcing of the infection rate

$$\beta(t) = \beta_0 \cdot (1 + \eta \cdot cos(\omega \cdot t)\ )$$

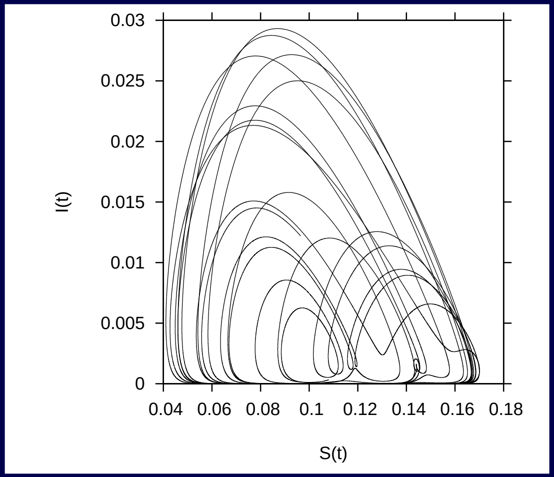
SIR system with seasonality



time series of $I(t)$

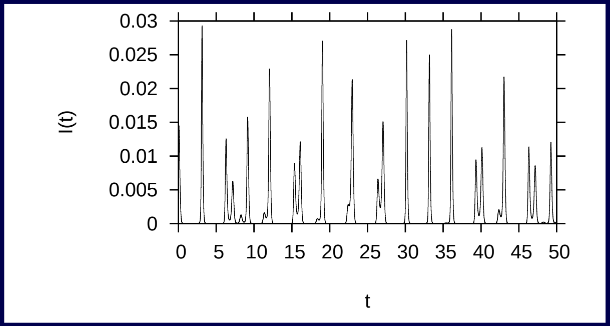


time series of $S(t)$

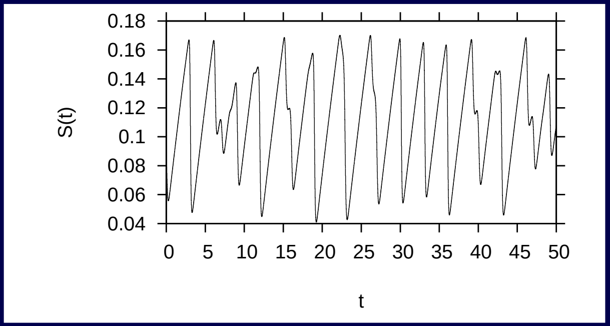


state space plot from time series
$I(t)$ versus $S(t)$

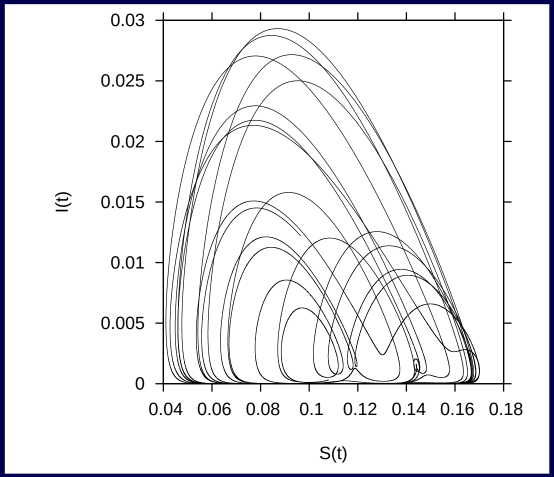
SIR system with seasonality



time series of $I(t)$

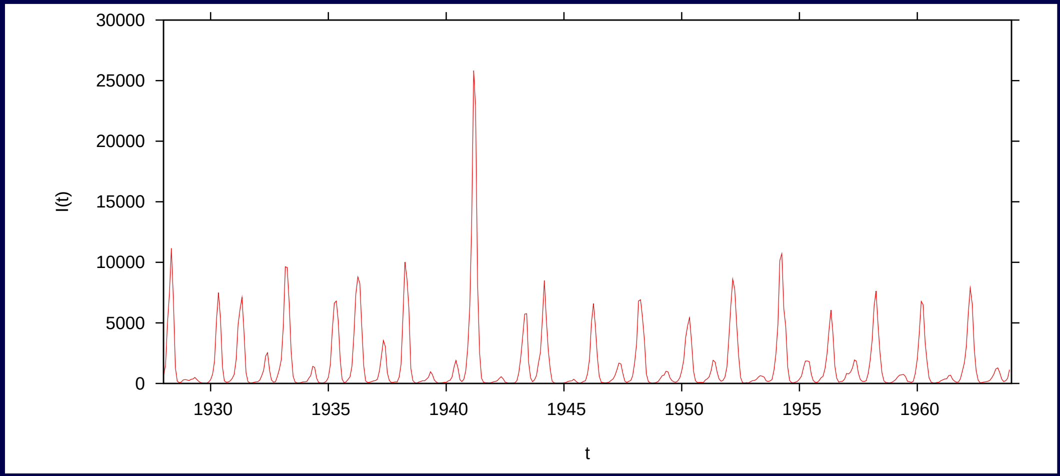


time series of $S(t)$

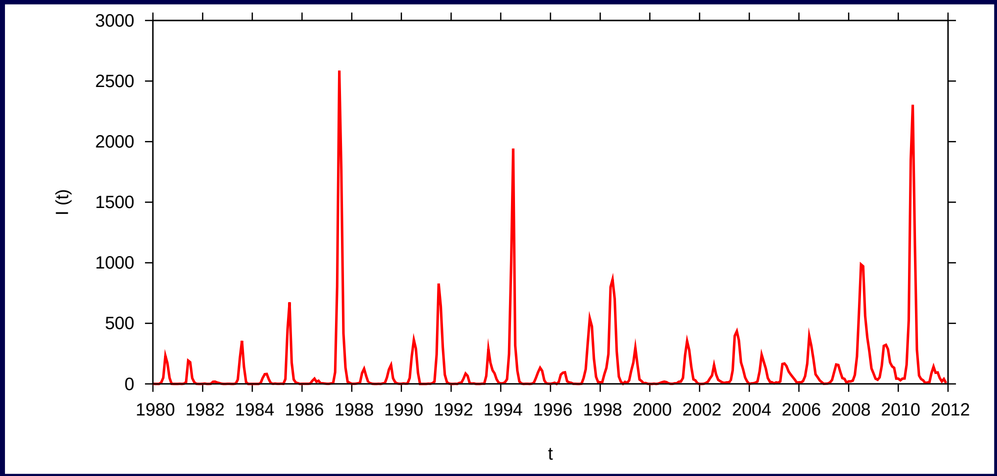


deterministic chaos

Epidemiological systems with various qualitative features



measles in New York City



dengue fever cases in Chiang Mai (Thailand)

Explicit multi-strain models:

example: dengue fever

Explicit multi-strain models:

example: dengue fever

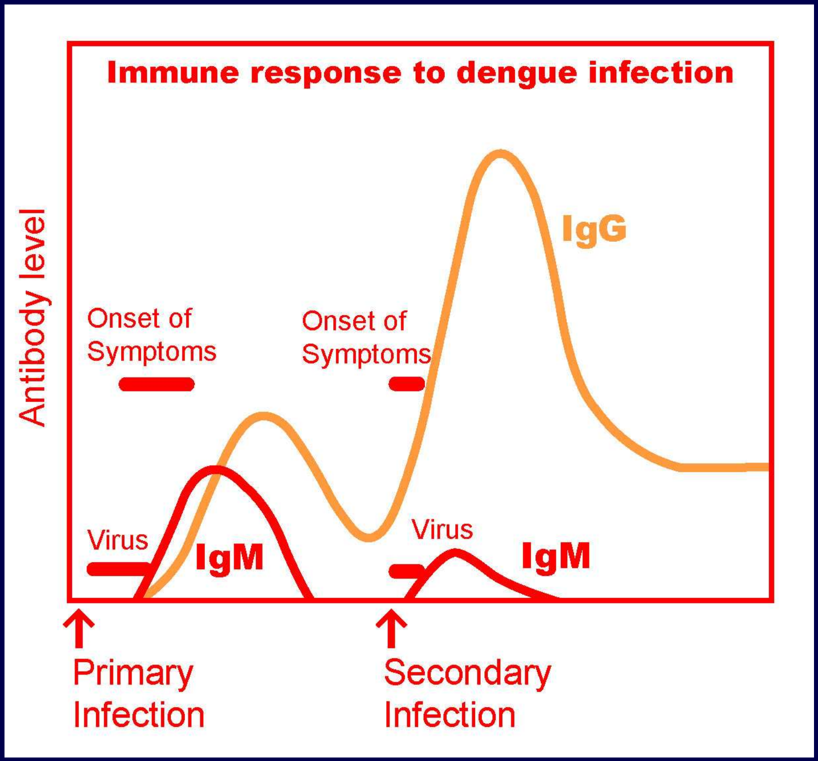
simplest example: two-strain SIR model
including antibody dependent enhancement (ADE)

=> chaos only for large ADE parameter ϕ

biologically motivated extension
including temporary cross immunity

=> chaos for much wider ϕ–region
(also for "inverse ADE")

Antibody dependent enhancement, ADE



... and temporary cross-immunity

Transition rates for two-strain SIR model with ADE and temporary cross immunity

$$
\begin{aligned}
S + I_1 &\xrightarrow{\beta_1} I_1 + I_1 \\
S + I_{21} &\xrightarrow{\phi_1 \beta_1} I_1 + I_{21} \\
I_1 &\xrightarrow{\gamma} R_1 \\
R_1 &\xrightarrow{\alpha} S_1 \\
S_1 + I_2 &\xrightarrow{\beta_2} I_{12} + I_2 \\
S_1 + I_{12} &\xrightarrow{\phi_2 \beta_2} I_{12} + I_{12} \\
I_{12} &\xrightarrow{\gamma} R
\end{aligned}
$$

Transition rates for two-strain SIR model with ADE and temporary cross immunity

$$S + I_1 \xrightarrow{\beta_1} I_1 + I_1$$

$$S + I_{21} \xrightarrow{\phi_1 \beta_1} I_1 + I_{21}$$

$$I_1 \xrightarrow{\gamma} R_1$$

$$R_1 \xrightarrow{\alpha} S_1$$

$$S_1 + I_2 \xrightarrow{\beta_2} I_{12} + I_2$$

$$S_1 + I_{12} \xrightarrow{\phi_2 \beta_2} I_{12} + I_{12}$$

$$I_{12} \xrightarrow{\gamma} R$$

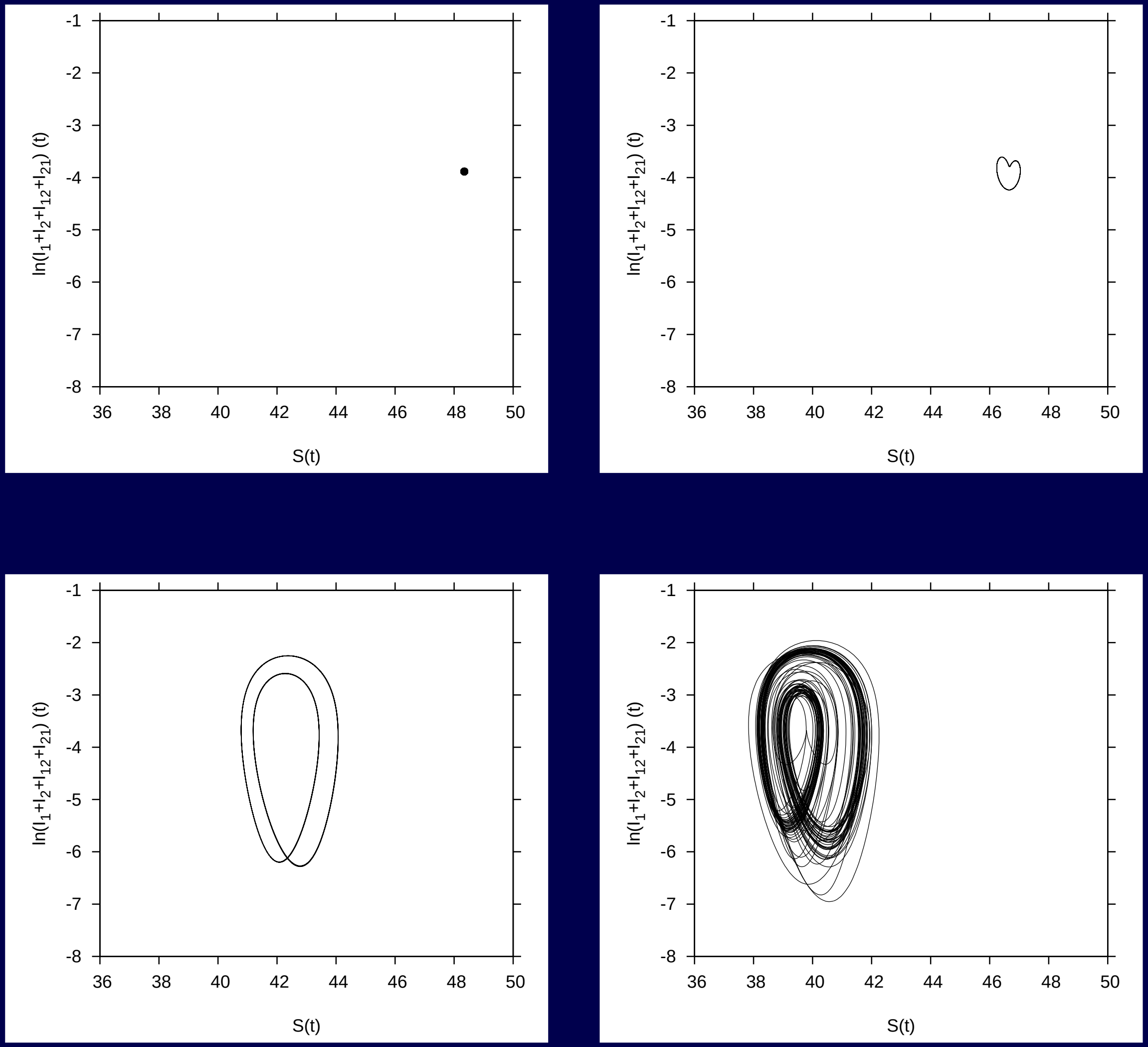
Transition rates for two-strain SIR model with ADE and temporary cross immunity

$$S + I_1 \xrightarrow{\beta_1} I_1 + I_1$$
$$S + I_{21} \xrightarrow{\phi_1\beta_1} I_1 + I_{21}$$
$$I_1 \xrightarrow{\gamma} R_1$$
$$R_1 \xrightarrow{\alpha} S_1$$
$$S_1 + I_2 \xrightarrow{\beta_2} I_{12} + I_2$$
$$S_1 + I_{12} \xrightarrow{\phi_2\beta_2} I_{12} + I_{12}$$
$$I_{12} \xrightarrow{\gamma} R$$

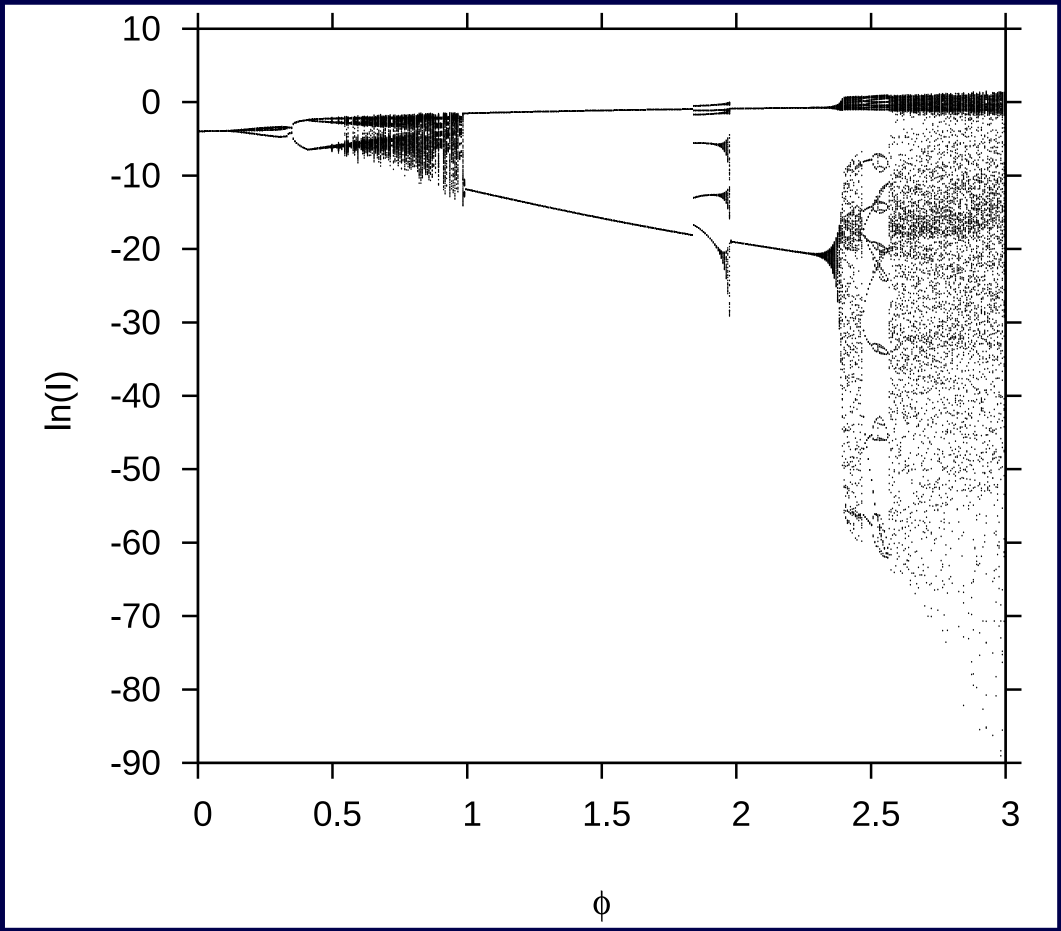
Multi-strain model for dengue fever

$$
\begin{aligned}
\frac{dS}{dt} &= -\frac{\beta_1}{N}S(I_1+\phi_1 I_{21}) - \frac{\beta_2}{N}S(I_2+\phi_2 I_{12}) + \mu(N-S)\\
\frac{dI_1}{dt} &= \frac{\beta_1}{N}S(I_1+\phi_1 I_{21}) - (\gamma+\mu)I_1\\
\frac{dI_2}{dt} &= \frac{\beta_2}{N}S(I_2+\phi_2 I_{12}) - (\gamma+\mu)I_2\\
\frac{dR_1}{dt} &= \gamma I_1 - (\alpha+\mu)R_1\\
\frac{dR_2}{dt} &= \gamma I_2 - (\alpha+\mu)R_2\\
\frac{dS_1}{dt} &= -\frac{\beta_2}{N}S_1(I_2+\phi_2 I_{12}) + \alpha R_1 - \mu S_1\\
\frac{dS_2}{dt} &= -\frac{\beta_1}{N}S_2(I_1+\phi_1 I_{21}) + \alpha R_2 - \mu S_2\\
\frac{dI_{12}}{dt} &= \frac{\beta_2}{N}S_1(I_2+\phi_2 I_{12}) - (\gamma+\mu)I_{12}\\
\frac{dI_{21}}{dt} &= \frac{\beta_1}{N}S_2(I_1+\phi_1 I_{21}) - (\gamma+\mu)I_{21}\\
\frac{dR}{dt} &= \gamma(I_{12}+I_{21}) - \mu R
\end{aligned}
$$

Bifurcations for changing ϕ



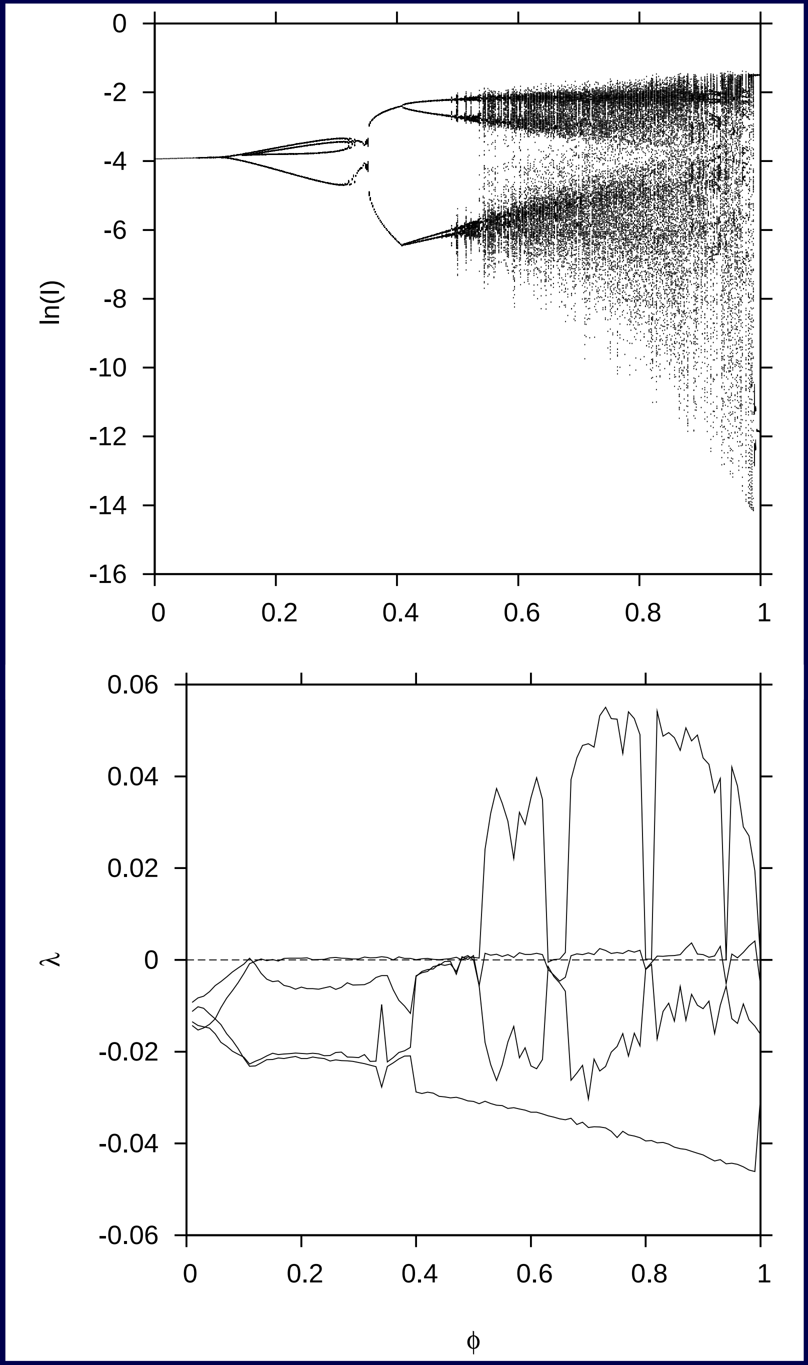
Bifurcation diagram



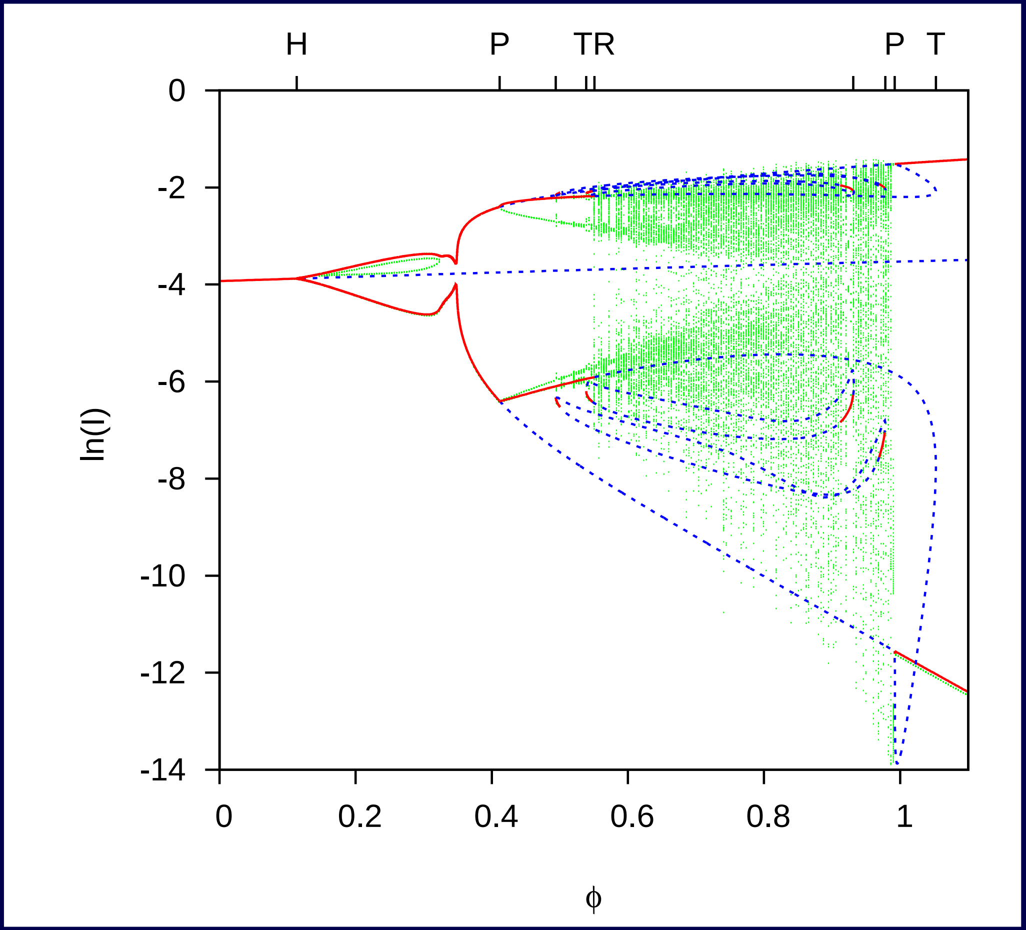
bifurcation diagram for $\alpha = 2$

i.e. $\frac{1}{2}$ year of temporary cross-immunity

Lyapunov spectrum versus bifurcation diagram



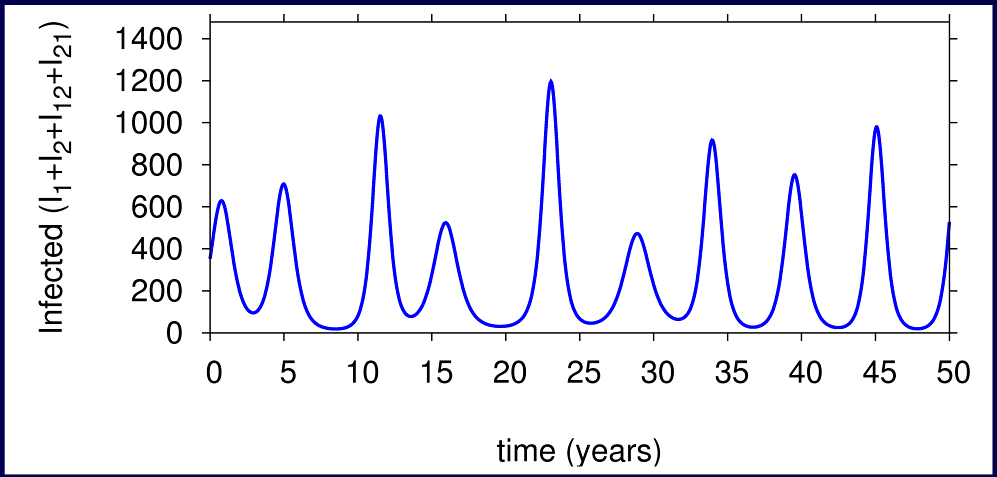
Bifurcation analysis via continuation: AUTO



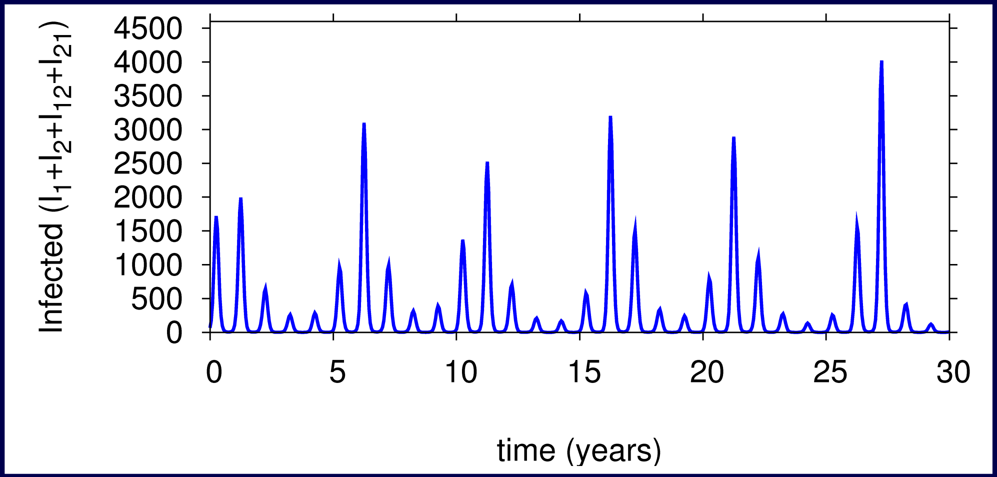
Comparing AUTO, Lyapunov spectra and numerical bifurcation diagrams:

coexisting attractors found, isolas

Including seasonality gives time series comparable to data

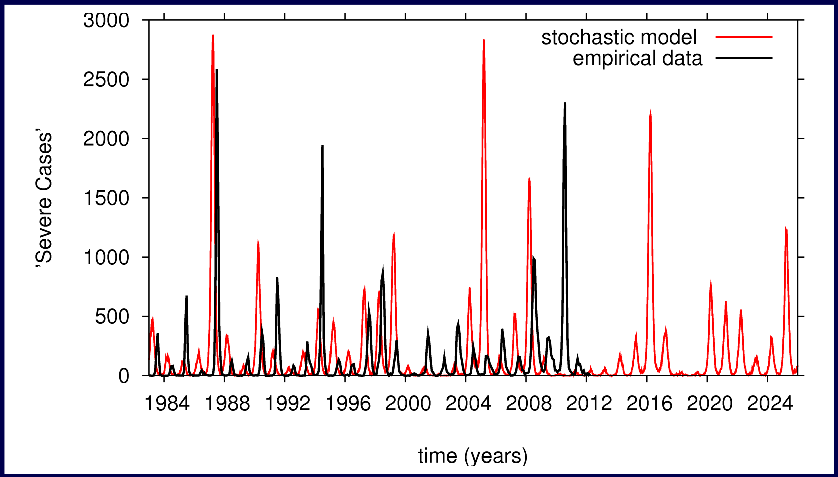


non-seasonal model in chaotic region

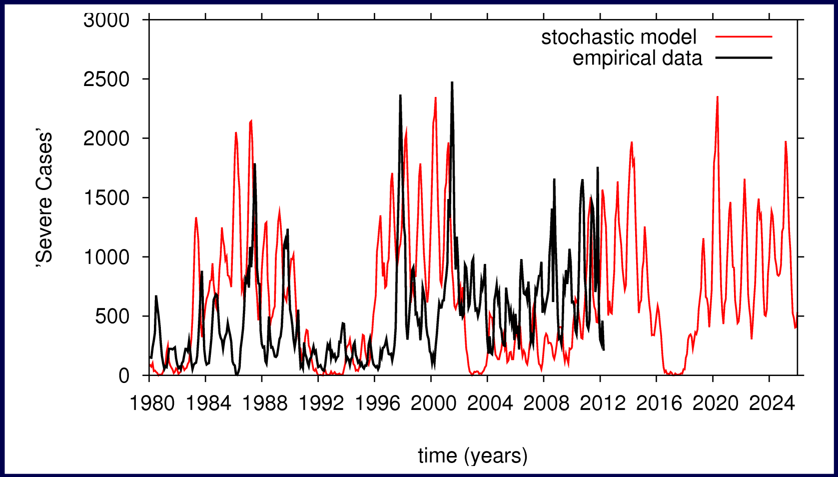


seasonality preserves chaotic pattern

Data matching: compare simulations with data

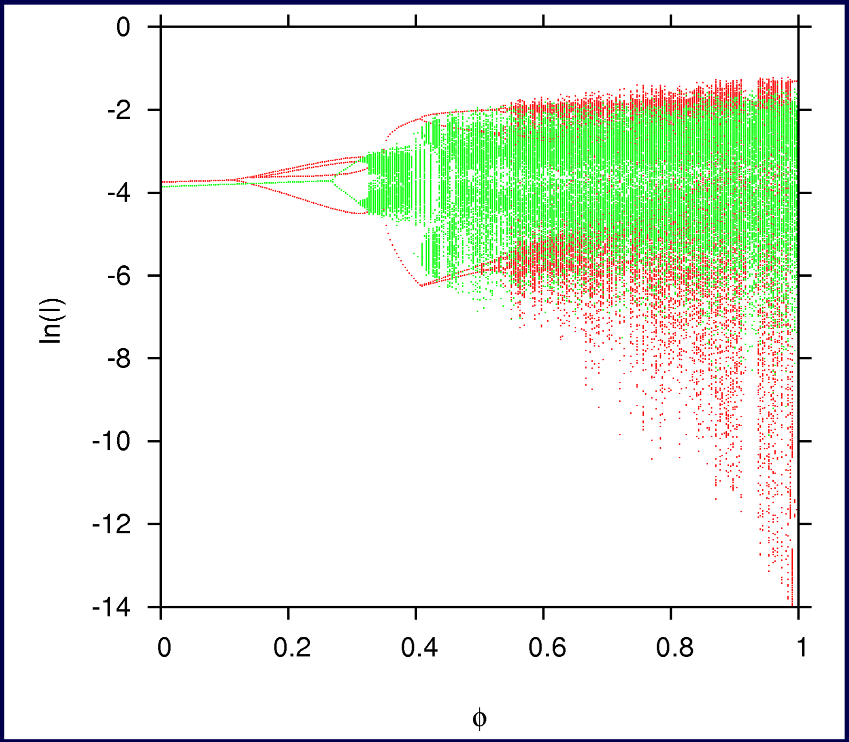


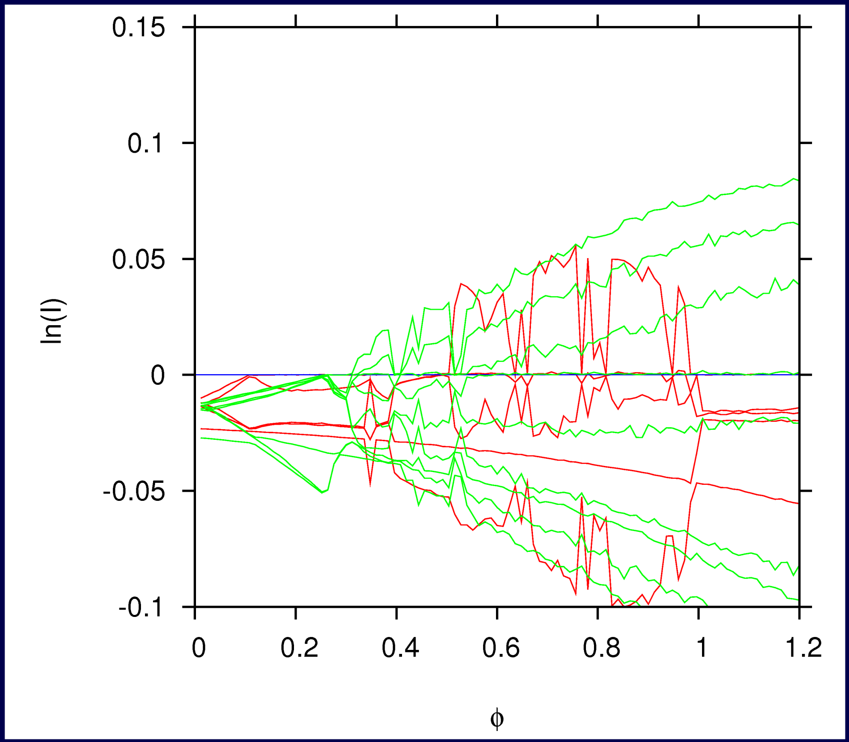
Chiang Mai



Krung Tep ("Bangkok")

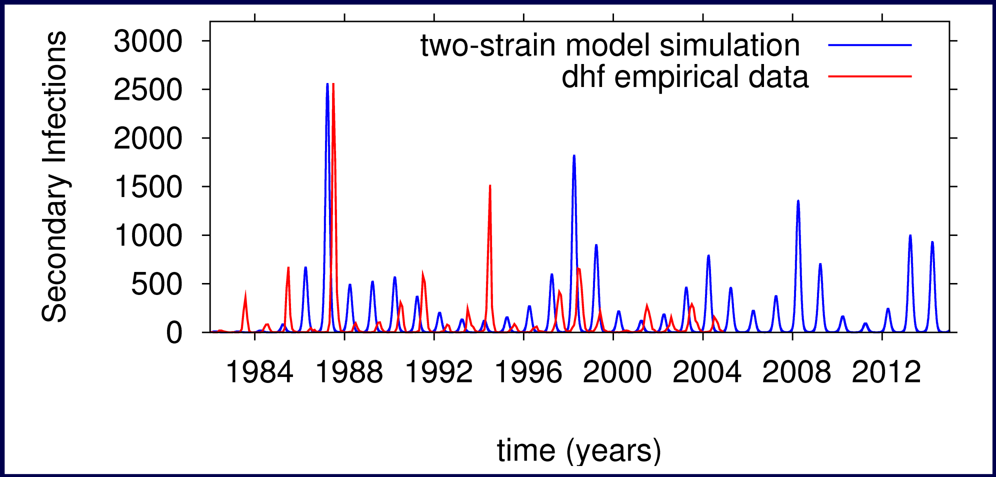
Comparing 2-strain with 4-strain models



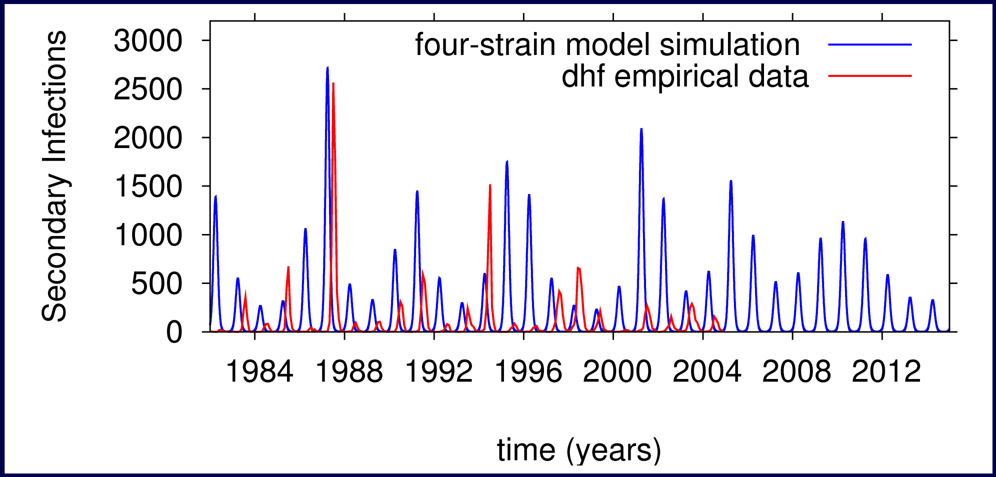


primary versus secondary infection drives the dynamics more than the exact number of strains

Model comparison:
2-strain versus 4-strain models



2-strain model versus Chiang Mai data



4-strain model versus Chiang Mai data

European Union project
DENFREE: "Dengue reasearch Framework for Resisting Epidemics in Europe"

DENFREE

Dengue Research Framework For
Resisting Epidemics In Europe
Since-2012

5 years project, start January 2012

together with 2 more EU project
"the largest financial effort on dengue research world wide"

European Union project
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DENFREE

Dengue Research Framework For
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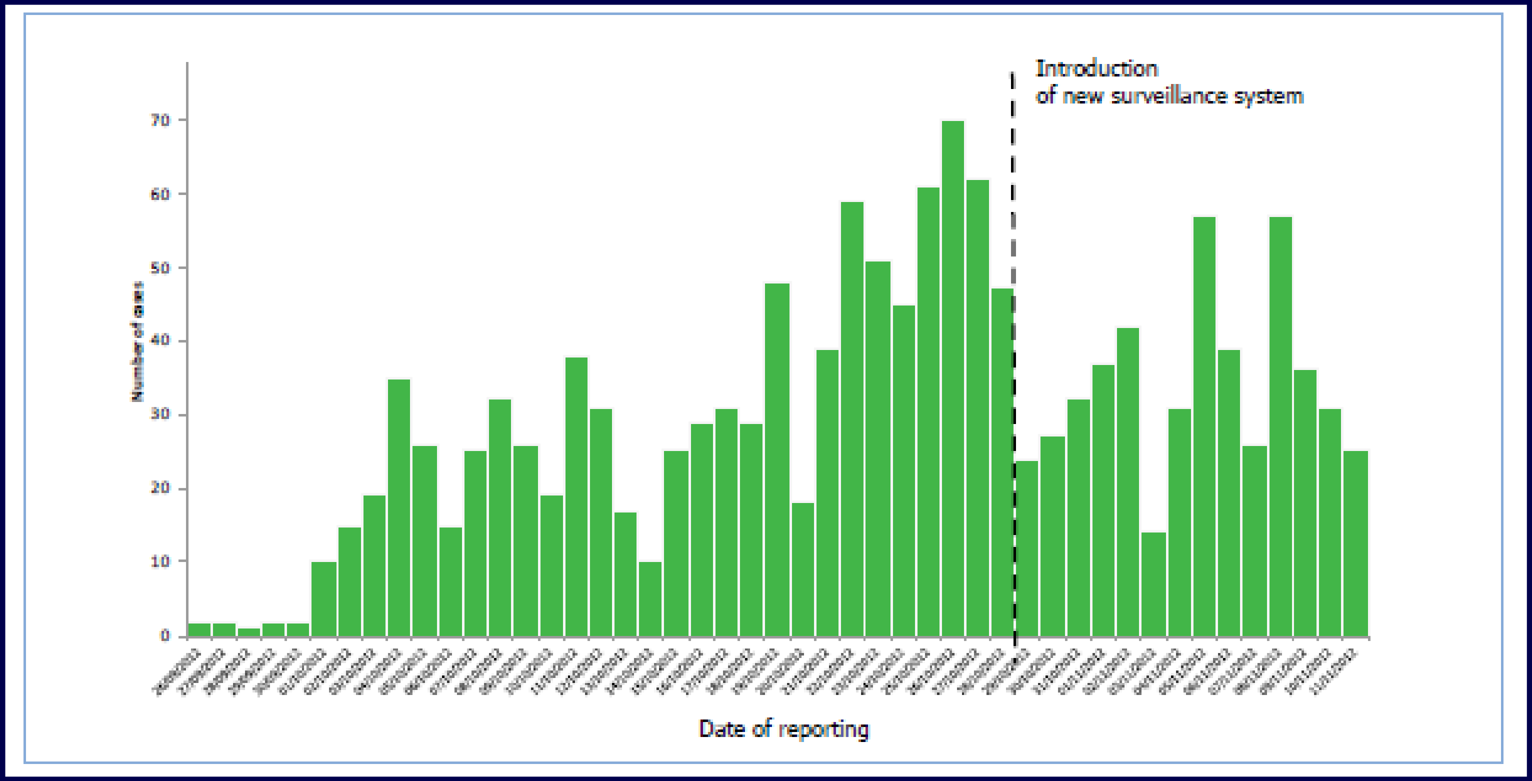
Since-2012

5 years project, start January 2012

together with 2 more EU project
"the largest financial effort on dengue research world wide"

CMAF is leading Work Package 4:
"Descriptic and predictive models for dengue fever"

Dengue fever outbreak on Madeira, Portugal, 2012
more than 2000 autochtonous cases detected



Dengue fever outbreak on Madeira, Portugal, 2012
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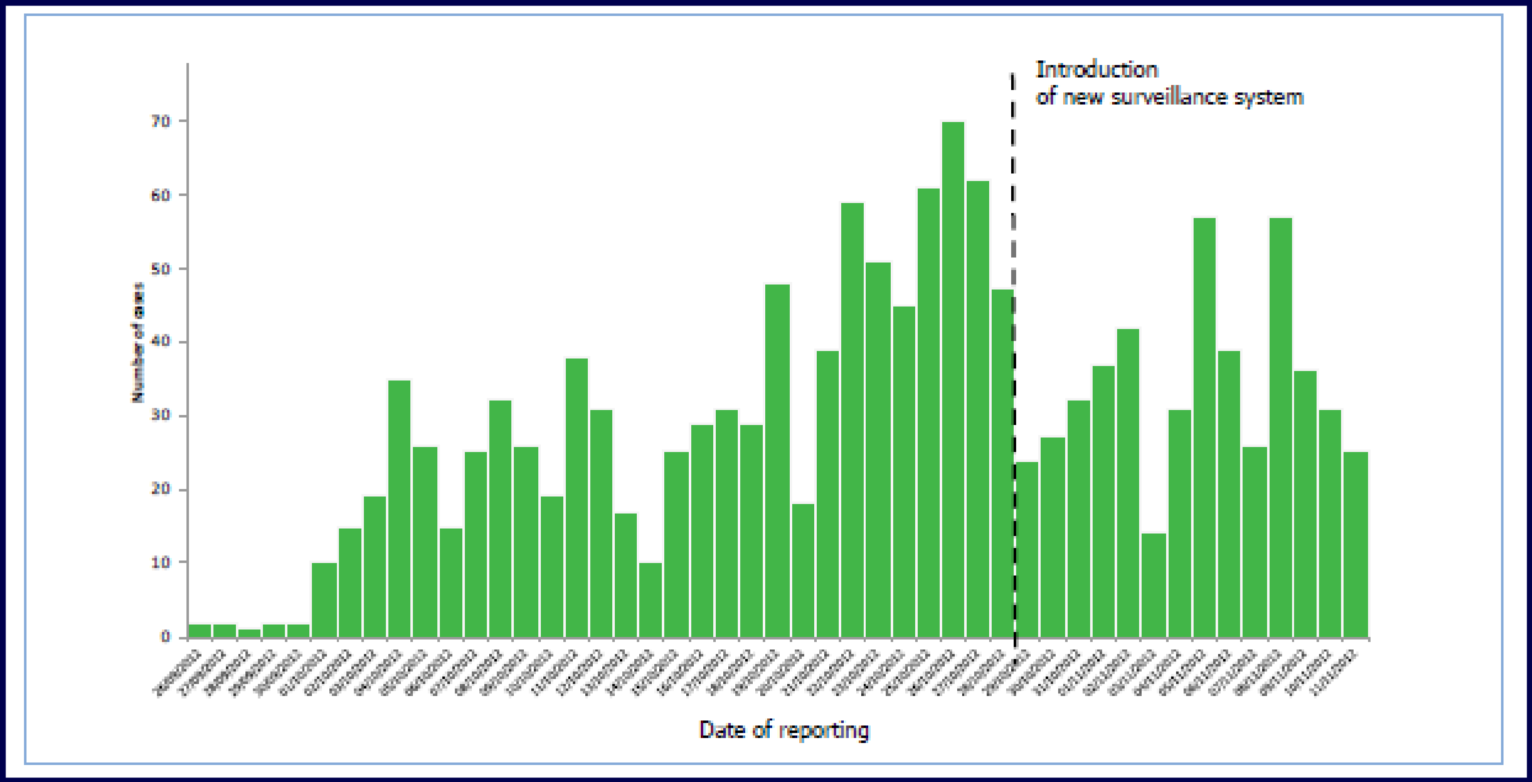


European Center for Disease Control (ECDC):
"The largest dengue outbreak in Europe
since the 1920th in Greece"

For data analysis:
stochastic modelling

Basic probability theory

joint probability

$$p(x, y)$$

marginal distribution

$$p(x) = \int p(x, y)\, dy$$

Bayes' rule

$$p(x, y) = p(x|y) \cdot p(y)$$

distribution that an event x_0 is given with certainty is $p(x) = \delta(x - x_0)$ with Dirac's delta-function

$$\int_a^b f(x) \cdot \delta(x - x_0)\, dx = f(x_0)$$

for x_0 between a and b

Application to epidemic processes

joint probability to find I_{n+1} infected at time $t + \Delta t$ and I_n at t

$$p(I_{n+1}, I_n)$$

marginal distribution to find only one of the variables no matter what the other variable does

$$p(I_{n+1}) = \sum_{I_n=0}^{N} p(I_{n+1}, I_n)$$

Bayes' rule gives conditional probability $p(I_{n+1}|I_n)$ for I_{n+1} knowing for sure I_n times $p(I_n)$

$$p(I_{n+1}, I_n) = p(I_{n+1}|I_n) \cdot p(I_n)$$

giving a dynamic evolution equation for probabilities of infected $p(I_n)$ at time t into $p(I_{n+1})$ at time $t + \Delta t$

$$p_{t+\Delta t}(I_{n+1}) = \sum_{I_n=0}^{N} p(I_{n+1}|I_n) \cdot p_t(I_n)$$

Application to epidemic processes

equation

$$p_{t+\Delta t}(I_{n+1}) = \sum_{I_n=0}^{N} p(I_{n+1}|I_n) \cdot p_t(I_n)$$

is a Perron-Frobenius type equation, and defines a time discrete Markov process

Application to epidemic processes

differential quotient gives time continous Markov process

$$\frac{p_{t+\Delta t}(I) - p_t(I)}{\Delta t} \approx \frac{d}{dt}\, p(I)$$

hence inserting time discrete version with $I := I_{n+1}$ and $\tilde{I} := I_n$

$$\frac{p_{t+\Delta t}(I) - p_t(I)}{\Delta t} = \sum_{\tilde{I}=0}^{N} \left(\frac{1}{\Delta t}\, p(I|\tilde{I}) \right) p_t(\tilde{I}) - \frac{1}{\Delta t}\, p_t(I)$$

and inserting normalization of conditioned probability $\sum_{\tilde{I}=0}^{N} p(\tilde{I}|I) = 1$ into the last term gives

$$\frac{d}{dt}\, p(I) = \sum_{\tilde{I}=0}^{N} w_{I|\tilde{I}}\, p_t(\tilde{I}) - \sum_{\tilde{I}=0}^{N} w_{\tilde{I}|I}\, p_t(I)$$

with transition rates $w_{I|\tilde{I}} := \left(\frac{1}{\Delta t}\, p(I|\tilde{I}) \right)$

Application to epidemic processes

equation

$$\frac{d}{dt}\,p(I) = \sum_{\tilde{I}=0,\tilde{I}\neq I}^{N} w_{I|\tilde{I}}\,p_t(\tilde{I}) - \sum_{\tilde{I}=0,\tilde{I}\neq I}^{N} w_{\tilde{I}|I}\,p_t(I)$$

is also called master equation and defines a time continuous state discrete Markov process

SIS epidemic

stochastic process

$$S + I \xrightarrow{\beta} I + I$$
$$I \xrightarrow{\alpha} S$$

for variable I and $S = N - I$ => probab. $p(I, t)$

$$\frac{d}{dt}\, p(I,t) \;=\; \frac{\beta}{N}(I-1)(N-(I-1))\, p(I-1,t) + \alpha(I+1)\, p(I+1,t) - \left(\frac{\beta}{N} I(N-I) + \alpha I\right)\, p(I,t)$$

mean $\langle I\rangle := \sum_{I=0}^{N} I \cdot p(I,t)$

$$\frac{d}{dt}\;\langle I\rangle = (\beta-\alpha)\langle I\rangle - \frac{\beta}{N}\langle I^2\rangle$$

and only in mean field approx. $var := \langle I^2\rangle - \langle I\rangle^2 \approx 0$

$$\frac{d}{dt}\;\langle I\rangle = \frac{\beta}{N}\langle I\rangle(N-\langle I\rangle) - \alpha\langle I\rangle$$

we obtain closed ODE

SIR epidemic

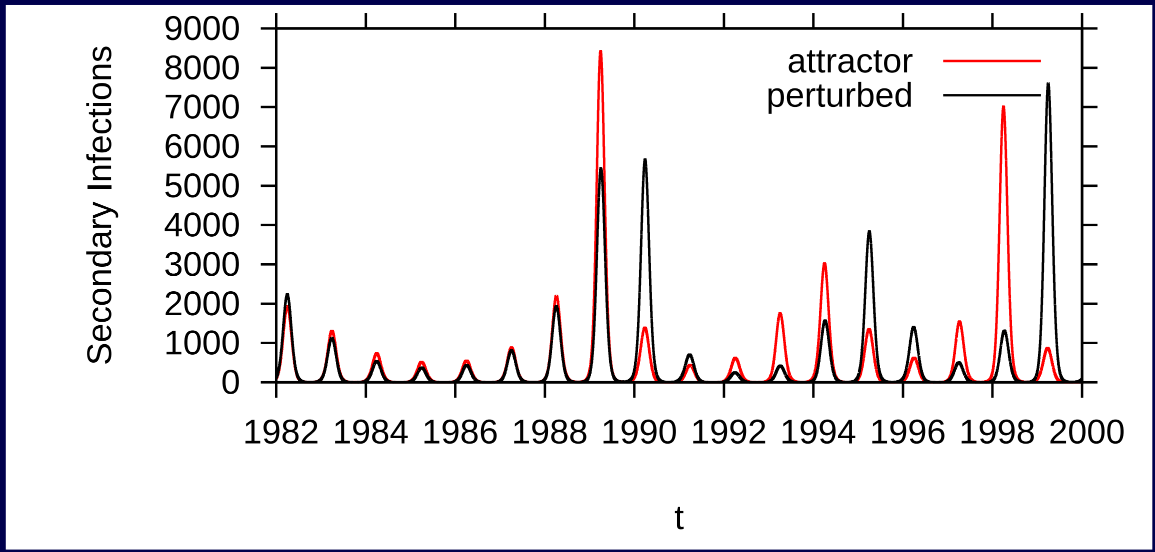
stochastic process

$$\begin{aligned} S + I &\xrightarrow{\beta} I + I \\ I &\xrightarrow{\gamma} R \\ R &\xrightarrow{\alpha} S \end{aligned}$$

for variables S, I and $R = N - S - I$ => probab. $p(S, I, t)$

$$\begin{aligned} \frac{d}{dt}\, p(S,I,t) \;=\; & \frac{\beta}{N}(I-1)(S+1)\; p(S+1, I-1, t) \\ & +\gamma(I+1)\; p(S, I+1, t) \\ & +\alpha(N-(S+1)-I)\; p(S+1, I, t) \\ & -\left(\frac{\beta}{N}SI + \gamma I + \alpha(N-S-I)\right)\; p(S,I,t) \end{aligned}$$

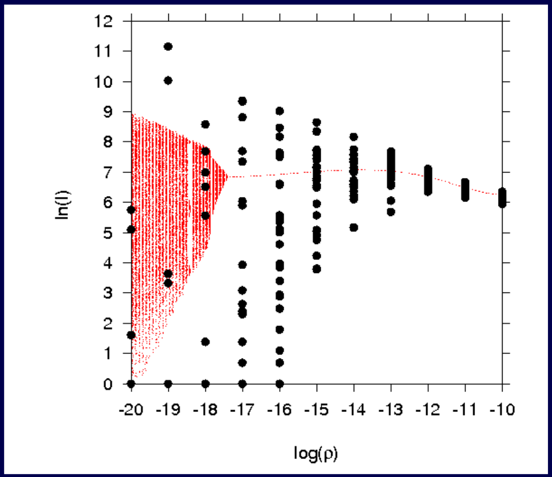
Short term predictability, long term unpredictability



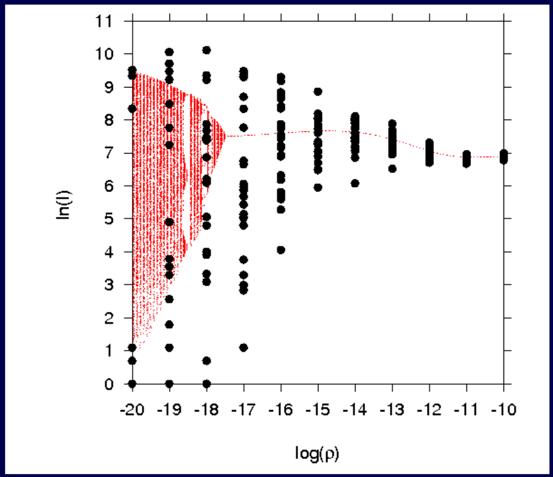
simulations with different initial conditions

implications for data analysis: Maximum Likelihood Iterated Filtering (MIF) is choice for such systems (Ionides et al 2006/ Bretó et al. 2009)

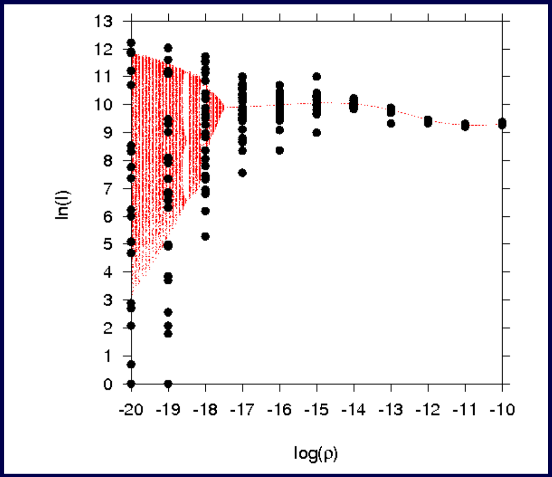
Parameter estimation in dengue: scaling with noise, importance of import



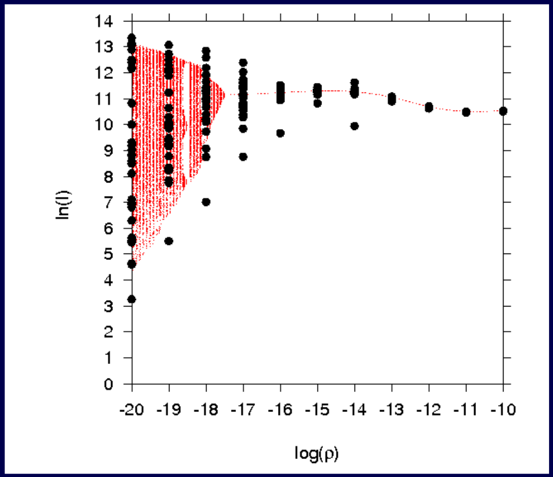
Chiang Mai
$N \approx 1$ mio.



North
$N \approx 6$ mio



Thailand
$N \approx 60$ mio.



South East Asia
$N \approx 250$ mio

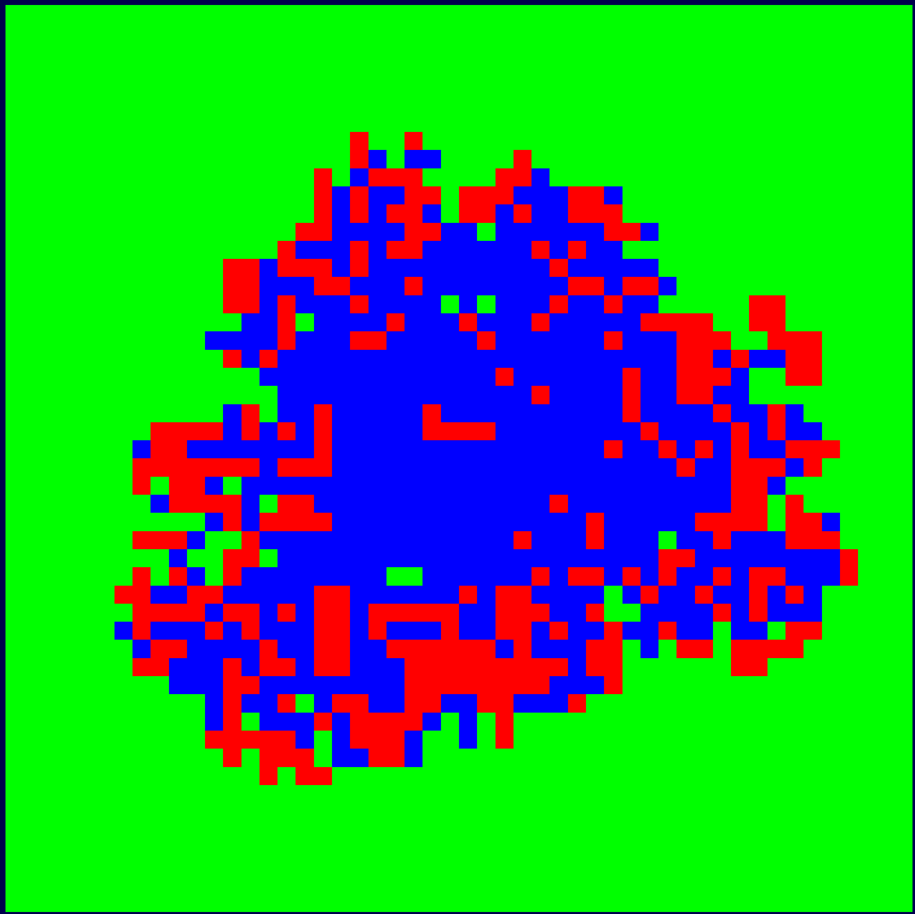
Individual based models

Individual based models
basic epidemic model: SIR

$$\begin{aligned} S_i + I_j &\xrightarrow{\beta} I_i + I_j \\ I_i &\xrightarrow{\gamma} R_i \\ R_i &\xrightarrow{\alpha} S_i \end{aligned}$$

Individual based models
example: stochastic 2 dimensional SIR epidemic

S green, I red, R blue



Individual based models
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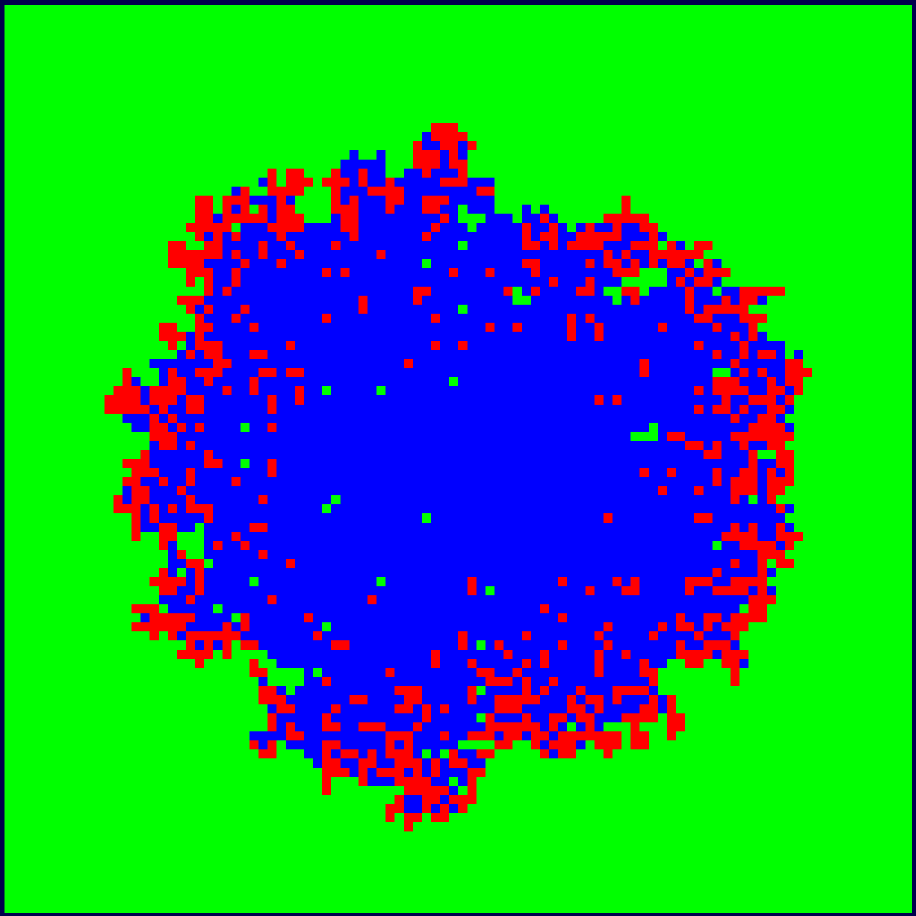

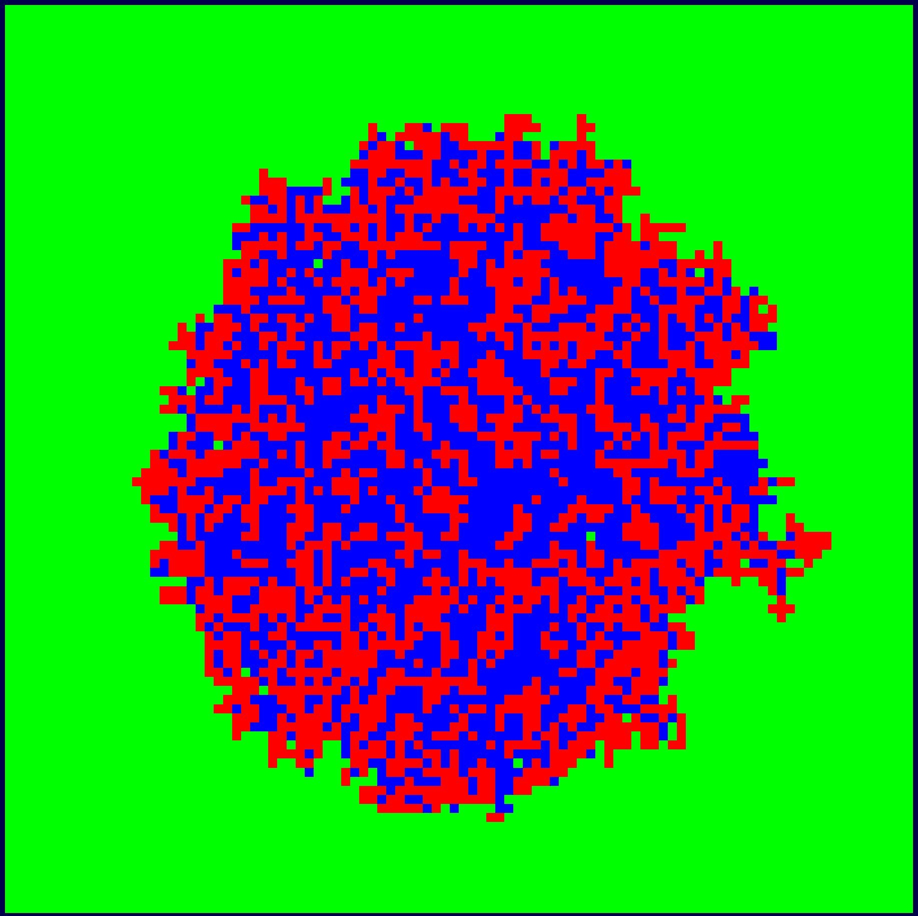

Master equation for spatial SIS model

consider as example again SIS epidemic model

$$S_i + I_j \xrightarrow{\beta} I_i + I_j$$
$$I_i \xrightarrow{\alpha} S_i$$

now at lattice site $i \in \{1, ..., N\}$ an infected $I_i = 1$, or not $I_i = 0$ hence $S_i := 1 - I_i = 1$, stochastic dynamics now given for variable $I_i \in \{0, 1\}$

Master equation for spatial SIS model

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stochastic dynamics now given for variables $I_i \in \{0, 1\}$ for $i \in \{1, ..., N\}$

Master equation for spatial SIS model

stochastic dynamics now given for variables $I_i \in \{0,1\}$ for $i \in \{1, ..., N\}$

$$
\begin{aligned}
\frac{d}{dt}\, p\, (I_1, I_2, ..., I_N, t) \\
&= \sum_{i=1}^{N} \beta \left(\sum_{j=1}^{N} J_{ij} I_j \right) I_i \;\; p(I_1, ..., 1 - I_i, ..., I_N, t) \\
&\quad + \sum_{i=1}^{N} \alpha (1 - I_i) \;\; p(I_1, ..., 1 - I_i, ..., I_N, t) \\
&\quad - \sum_{i=1}^{N} \left[\beta \left(\sum_{j=1}^{N} J_{ij} I_j \right) (1 - I_i) + \alpha I_i \right] \;\; p(I_1, ..., I_i, ..., I_N, t)
\end{aligned}
$$

with adjacency matrix $J_{ij} \in \{0,1\}$

Clusters and their dynamics

total number of infected individuals on the lattics

$$[I] := \sum_{i=1}^{N} I_i$$

total number of susceptibles

$$[S] := \sum_{i=1}^{N} (1 - I_i)$$

total number of pairs

$$[II] := \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} \; I_i \cdot I_j$$

triples

$$[III] := \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} J_{ij} J_{jk} \cdot I_i I_j I_k$$

Clusters and their dynamics

triangles

$$[\Delta] := \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} J_{ij} J_{jk} J_{ki} \cdot I_i I_j I_k$$

space averages, e.g. $[I] := \sum_{i=1}^{N} I_i$, depend on the ensemble $(I_1, ..., I_N)$, hence define ensemble average

$$\langle I \rangle(t) := \sum_{I_1=0}^{1} ... \sum_{I_N=0}^{1} [I]\, p(I_1, ..., I_N, t)$$

and generally for any function $f = f(I_1, ..., I_N)$

$$\langle f \rangle(t) := \sum_{I_1=0}^{1} ... \sum_{I_N=0}^{1} f(I_1, ..., I_N)\, p(I_1, ..., I_N, t)$$

then time evolution to calculate from master equation

$$\frac{d}{dt}\langle f \rangle(t) := \sum_{I_1=0}^{1} ... \sum_{I_N=0}^{1} f(I_1, ..., I_N)\, \frac{d}{dt} p(I_1, ..., I_N, t)$$

Dynamics of expectation values in spatial systems

local mean value dynamics

$$\frac{d}{dt}\langle I_i\rangle := \sum_{I_1=0}^{1} \dots \sum_{I_N=0}^{1} I_i \frac{d}{dt} p(I_1, \dots, I_N, t)$$

$$= \dots$$

$$= \beta \sum_{j=1}^{N} J_{ij}\langle I_j(1-I_i)\rangle - \alpha\langle I_i\rangle$$

$$= \beta \sum_{j=1}^{N} J_{ij}\langle S_i I_j\rangle - \alpha\langle I_i\rangle$$

Dynamics of expectation values in spatial systems

global mean value dynamics

$$\frac{d}{dt}\langle I\rangle = \sum_{i=1}^{N}\frac{d}{dt}\langle I_i\rangle$$

$$= b\left(Q\langle I\rangle - \langle II\rangle_1\right) - a\langle I\rangle$$

$$= b\langle SI\rangle_1 - a\langle I\rangle$$

using $Q_i := \sum_{j=1}^{N} J_{ij}$ number of sites connected to site i, and for regular lattices $Q_i = Q$ constant

Dynamics of expectation values in spatial systems

global mean value dynamics

$$\frac{d}{dt}\langle I\rangle = b\left(Q\langle I\rangle - \langle II\rangle_1\right) - a\langle I\rangle$$

$$= b\langle SI\rangle_1 - a\langle I\rangle$$

contains pair expectations $\langle II\rangle_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\langle I_i I_j\rangle$
hence to calculate dynamics for pairs

$$\frac{d}{dt}\langle II\rangle_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\frac{d}{dt}\langle I_i I_j\rangle$$

giving

$$\frac{d}{dt}\langle II\rangle_1 = 2b\left(\langle II\rangle_2 - \langle III\rangle_{1,1}\right) - 2a\langle II\rangle_1$$

$$= 2b\langle ISI\rangle_{1,1} - 2a\langle II\rangle_1$$

Dynamics of expectation values in spatial systems

dynamics for pairs

$$\frac{d}{dt}\langle II\rangle_1 = 2b\left(\langle II\rangle_2 - \langle III\rangle_{1,1}\right) - 2a\langle II\rangle_1$$

$$= 2b\langle ISI\rangle_{1,1} - 2a\langle II\rangle_1$$

now includes triples

$$\langle ISI\rangle_{1,1} := \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} J^1_{ij}J^1_{jk}\langle I_i(1-I_j)I_k\rangle$$

or longer range correlations

$$\langle II\rangle_2 := \sum_{i=1}^{N}\sum_{k=1}^{N}\underbrace{\left(\sum_{j=1}^{N} J_{ij}\, J_{jk}\right)}_{=:(J^2)_{ik}}\langle I_i I_k\rangle$$

Approximating pairs

global mean value dynamics contains pairs

$$\frac{d}{dt}\langle I\rangle = b\left(Q\langle I\rangle - \langle II\rangle_1\right) - a\langle I\rangle$$

which can be approximated by an interaction of I_i not with its neigbours I_j but with an average of all other sites $\langle I\rangle$

$$\sum_{j=1}^{N} J_{kj}I_j \approx \sum_{j=1}^{N} J_{kj}\,\frac{\langle I\rangle}{N}$$

called "interacting with a mean field" instead of its neighbours directly

Approximating pairs

global mean value dynamics contains pairs

$$\frac{d}{dt} \langle I \rangle = b \left(Q \langle I \rangle - \langle II \rangle_1 \right) - a \langle I \rangle$$

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$$\sum_{j=1}^{N} J_{kj} I_j \approx \sum_{j=1}^{N} J_{kj} \frac{\langle I \rangle}{N}$$

called "interacting with a mean field" instead of its neighbours directly, hence

mean field approximation

Mean field approximation

technically

$$
\begin{aligned}
\langle II \rangle_1 \;&=\; \langle \sum_{i=1}^{N} \sum_{j=1}^{N} \; J_{ij} I_i I_j \rangle = \langle \sum_{i=1}^{N} \; I_i \; \sum_{j=1}^{N} \; J_{ij} I_j \rangle \\
&\approx\; \langle \sum_{i=1}^{N} \; I_i \, \frac{Q}{N} \cdot \langle I \rangle \rangle = \frac{Q}{N} \cdot \langle I \rangle \cdot \langle \sum_{i=1}^{N} \; I_i \; \rangle \\
&=\; \frac{Q}{N} \cdot \langle I \rangle^2
\end{aligned}
$$

giving in the dynamics of the mean a closed ODE

$$
\begin{aligned}
\frac{d}{dt} \langle I \rangle \;&=\; b \left(Q \langle I \rangle - \frac{Q}{N} \langle I \rangle^2 \right) - a \langle I \rangle \\
&=\; \frac{bQ}{N} \, \langle I \rangle \, \left(N - \langle I \rangle \right) - a \langle I \rangle
\end{aligned}
$$

Mean field approximation

technically

$$\langle II \rangle_1 = \langle \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} I_i I_j \rangle = \langle \sum_{i=1}^{N} I_i \sum_{j=1}^{N} J_{ij} I_j \rangle$$

$$\approx \langle \sum_{i=1}^{N} I_i \frac{Q}{N} \cdot \langle I \rangle \rangle = \frac{Q}{N} \cdot \langle I \rangle \cdot \langle \sum_{i=1}^{N} I_i \rangle$$

$$= \frac{Q}{N} \cdot \langle I \rangle^2$$

giving in the dynamics of the mean a closed ODE

$$\frac{d}{dt} \langle I \rangle = b \left(Q \langle I \rangle - \frac{Q}{N} \langle I \rangle^2 \right) - a \langle I \rangle$$

$$= \frac{bQ}{N} \langle I \rangle \left(N - \langle I \rangle \right) - a \langle I \rangle$$

our famous ODE of the SIS system from before
with $\beta = bQ$

Approximating triples into pairs

global dynamics dynamics contains triples

$$\frac{d}{dt}\langle II\rangle_1 = 2b\left(\langle II\rangle_2 - \langle III\rangle_{1,1}\right) - 2a\langle II\rangle_1$$

$$= 2b\langle ISI\rangle_{1,1} - 2a\langle II\rangle_1$$

idea of approximation

$$\langle SIR\rangle \approx \frac{\langle SI\rangle \cdot \langle IR\rangle}{\langle I\rangle}$$

is basically a Bayes' rule now for conditioned mean values $\langle I_i|I_j\rangle := \sum_{I_i=0}^{1} I_i\, p(I_i|I_j)$, namely

$$\langle I_i I_j\rangle = \langle I_i|I_j = 1\rangle \cdot \langle I_j\rangle$$

respecively for triples

$$\langle I_i I_j I_k\rangle = \langle I_i|I_j = 1, I_k = 1\rangle \cdot \langle I_j I_k\rangle$$

Approximating triples into pairs

essential approximation: the mean at location i depends on its neighbour j, but not on the next neighbour k

$$\langle I_i | I_j = 1, I_k = 1 \rangle \approx \langle I_i | I_j = 1 \rangle$$

giving

$$\langle I_i I_j I_k \rangle \approx \frac{\langle I_i I_j \rangle}{\langle I_j \rangle} \cdot \langle I_j I_k \rangle$$

and then to pull up to global quantities

Dynamics in pair approximation

from the original system

$$\frac{d}{dt}\langle I\rangle = b\,(Q\langle I\rangle - \langle II\rangle_1) - a\langle I\rangle$$

$$\frac{d}{dt}\,\langle II\rangle_1 = 2b\left(\langle II\rangle_2 - \langle III\rangle_{1,1}\right) - 2a\langle II\rangle_1$$

in pair approximation we obtain for densities

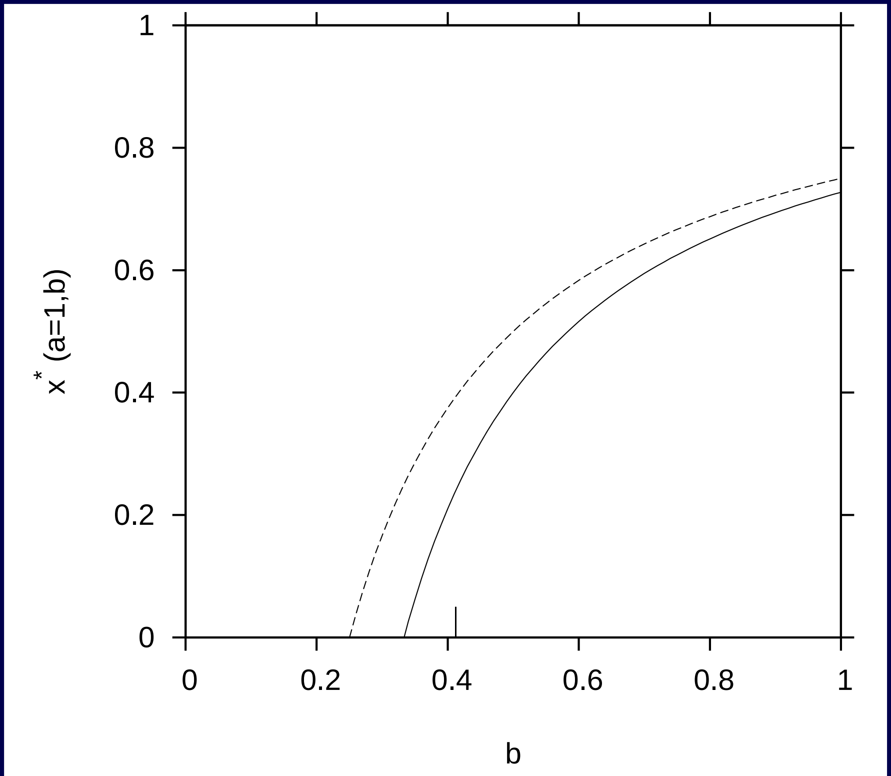
$$\begin{aligned} x &:= \langle I\rangle/N && \in [0,1] \\ y &:= \langle II\rangle_1/(NQ) && \in [0,1] \end{aligned}$$

the closed ODE system

$$\frac{dx}{dt} = bQ(x-y) - ax$$

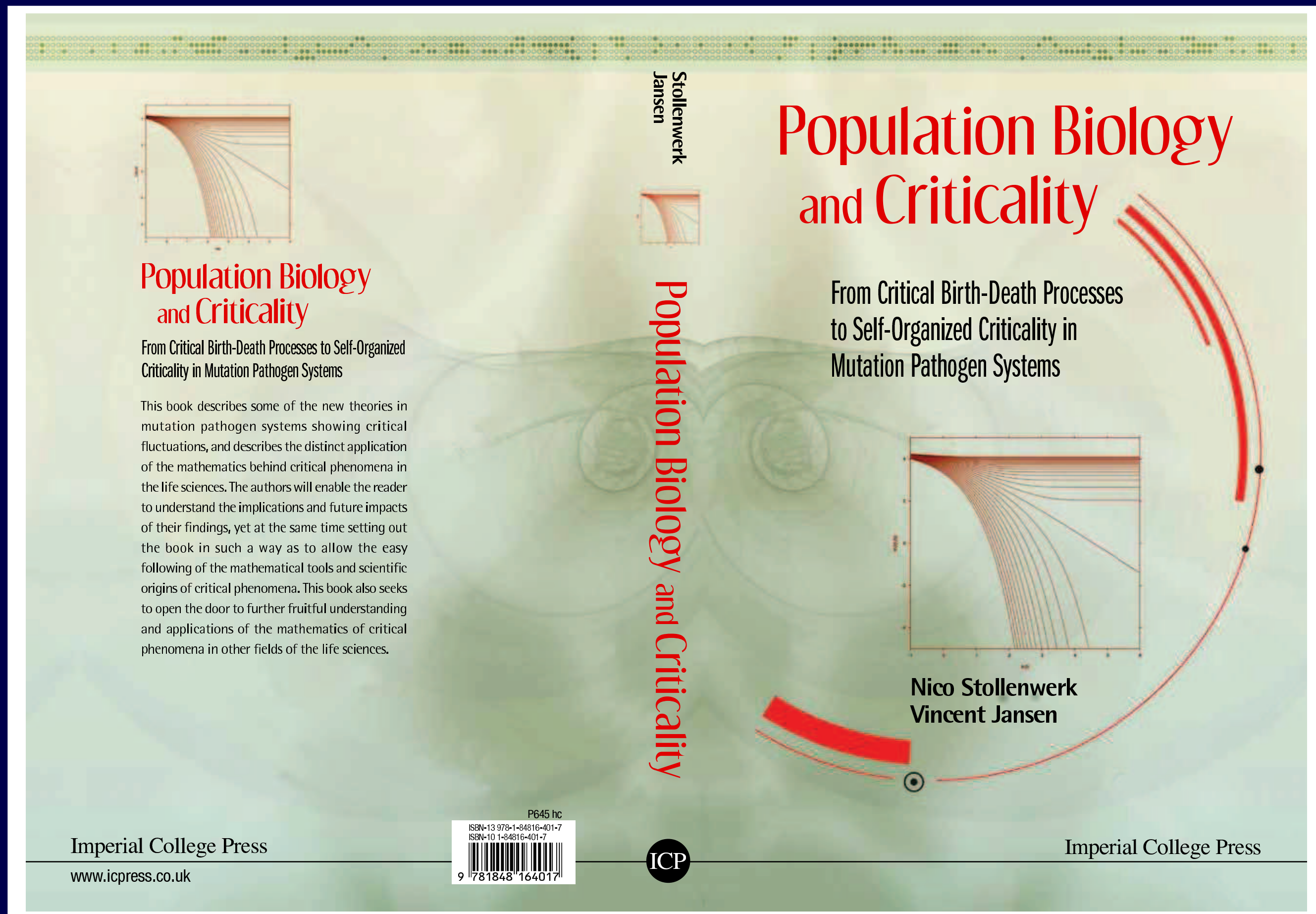
$$\frac{dy}{dt} = 2b(Q-1)\frac{(x-y)^2}{1-x} + 2b(x-y) - 2ay$$

Dynamics in pair approximation



mean field, pair approximation, spatial simulation (from left to right)

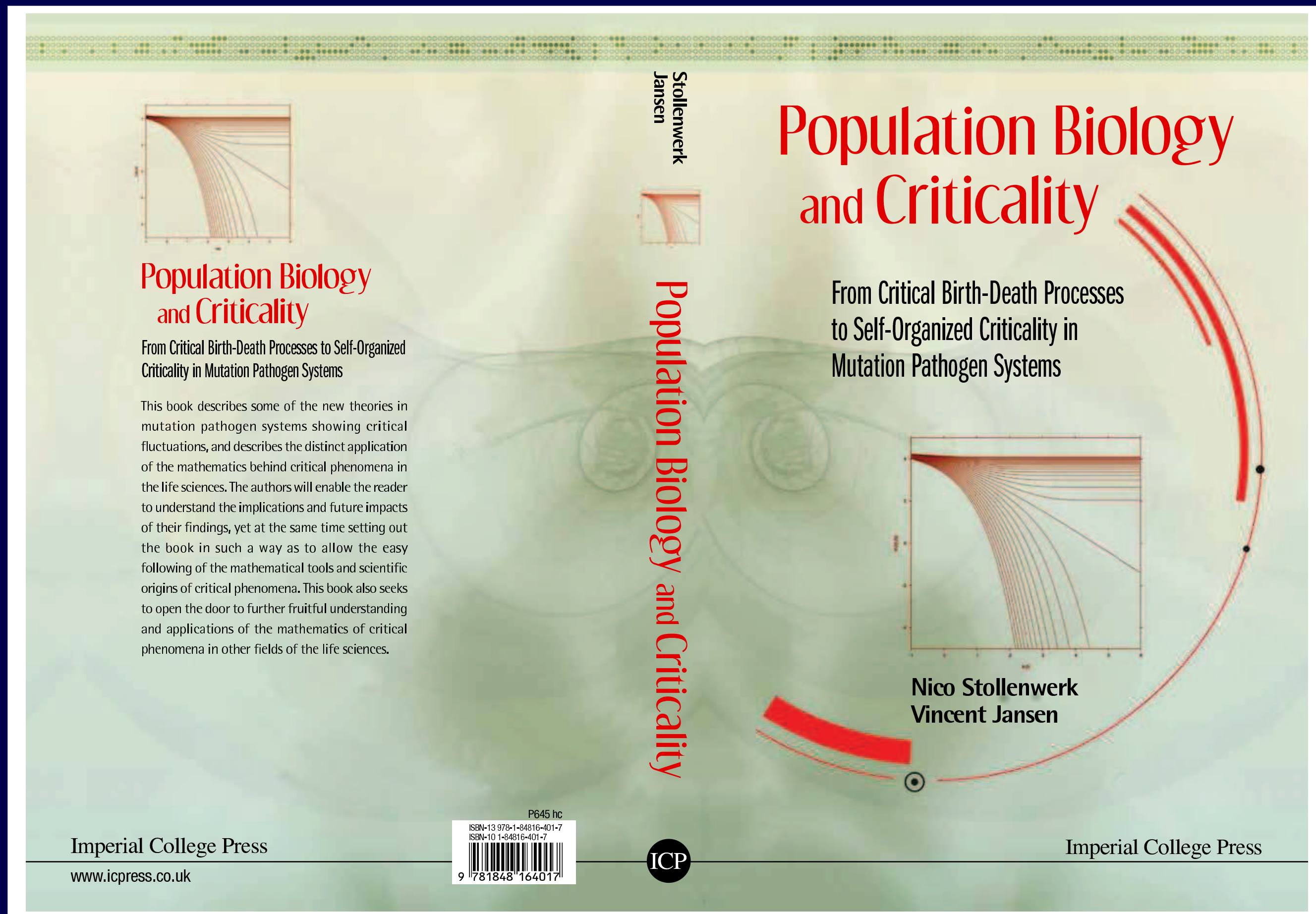
Theory of accidental pathogens, paradigmatic system: bacterial meningitis



in many-strain systems with fast evolution

evolution towards critical fluctuations (SOC :-)

Theory of accidental pathogens, paradigmatic system: bacterial meningitis



Muñoz et al. (2011): the "Stollenwerk-Jansen model (SJ)" is in universality class of voter model

Dynamics of expectation values
in spatial systems

local mean value dynamics

$$\frac{d}{dt}\langle I_i \rangle := \sum_{I_1=0}^{1} \dots \sum_{I_N=0}^{1} I_i \quad \frac{d}{dt} p(I_1, \dots, I_N, t)$$

$$= \dots$$

$$= \beta \sum_{j=1}^{N} J_{ij} \langle I_j (1 - I_i) \rangle - \alpha \langle I_i \rangle$$

$$= \beta \sum_{j=1}^{N} J_{ij} \langle S_i I_j \rangle - \alpha \langle I_i \rangle$$

Surrogate for human contact dynamics: exchanging money :-)

reaction scheme for exchanging an item from item holder I_i to susceptible S_j to receive this item

$$S_i + I_j \xrightarrow{\chi} I_i + S_j$$
$$I_i + S_j \xrightarrow{\chi} S_i + I_j$$

gives dynamics of local expectation value

$$\frac{d}{dt} \langle I_i \rangle = \chi \sum_{j=1}^{N} J_{ij} \left(\langle I_j \rangle - \langle I_i \rangle \right) =: \chi \cdot \Delta \langle I_i \rangle$$

diffusion equation for regular lattices, generalizable to contact probabilities proportional to distance of individuals

"spatially restricted networks"

Surrogate for human contact dynamics: exchanging money :-)

superdiffusion via fracional Laplace operator for $\mu \leq 2$

$$\frac{\partial}{\partial t}u(x,t) = \chi \frac{\partial^\mu}{\partial x^\mu}u(x,t)$$

defined via the Fourier transform (Riesz fractional derivative) $\frac{\partial^\mu}{\partial x^\mu}e^{ikx} := -|k|^\mu \cdot e^{ikx}$ in

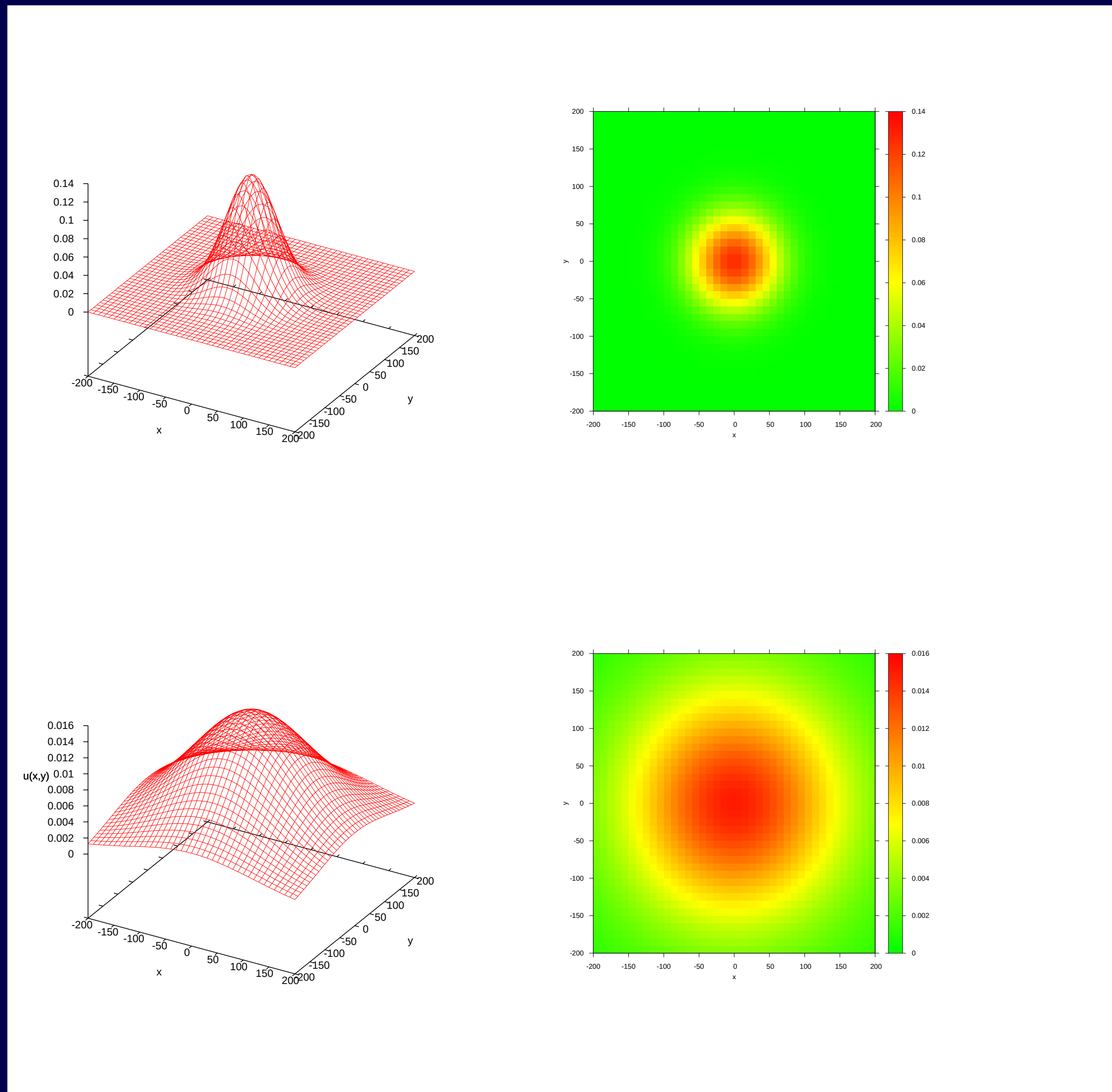
$$\frac{\partial}{\partial t}u(x,t) = \chi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k,t) \left(\frac{\partial^\mu}{\partial x^\mu} e^{ikx} \right) \, \mathrm{d}k$$

using convolution and with constant
$c_\mu := \frac{1}{\pi}\Gamma(\mu+1)\sin\left(\frac{\pi}{2}\mu\right)$

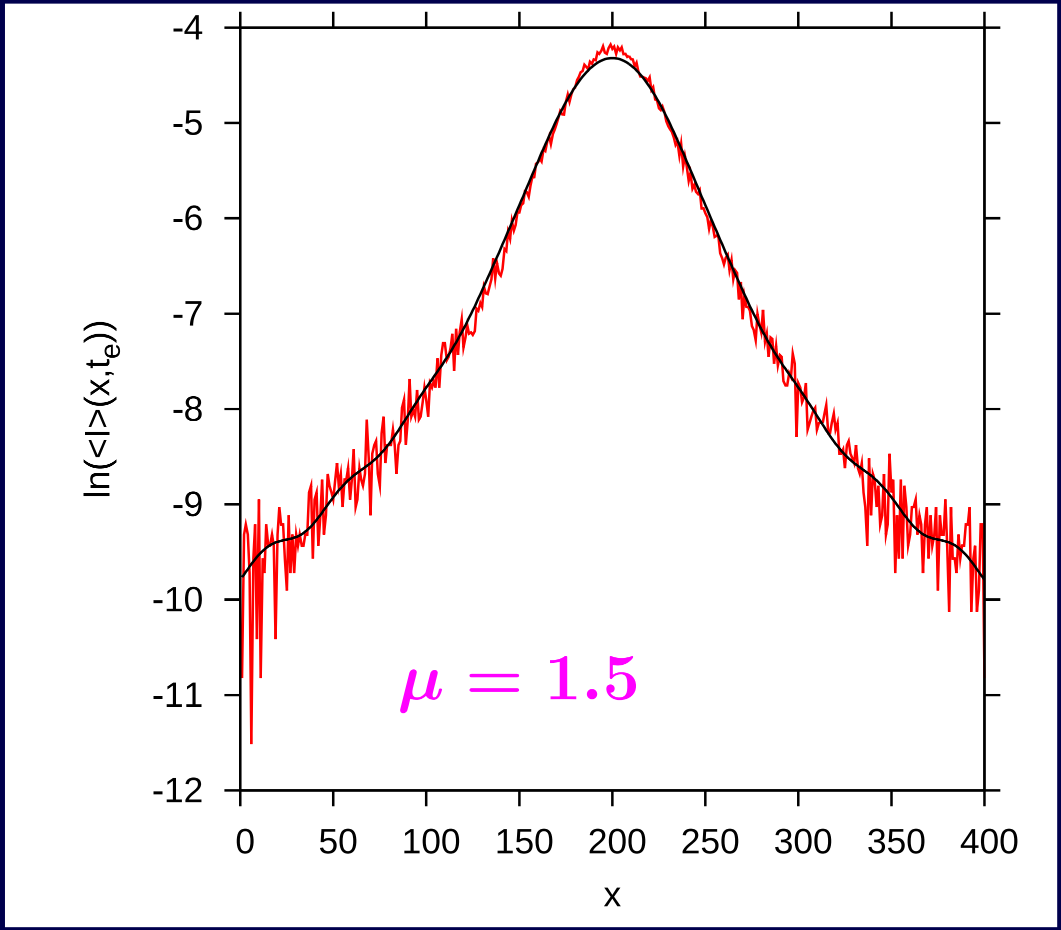
$$\frac{\partial}{\partial t}u(x,t) = (\chi \cdot c_\mu) \int_{-\infty}^{\infty} \frac{u(y,t)}{|x-y|^{\mu+1}} \mathrm{d}y$$

as integral representation of superdiffusive Laplace

Super-diffusion in higher dimensions



Spatial spreading in epidemiological systems: Superdiffusion using fractional calculus



stochastic histogram and fractional diffusion equation

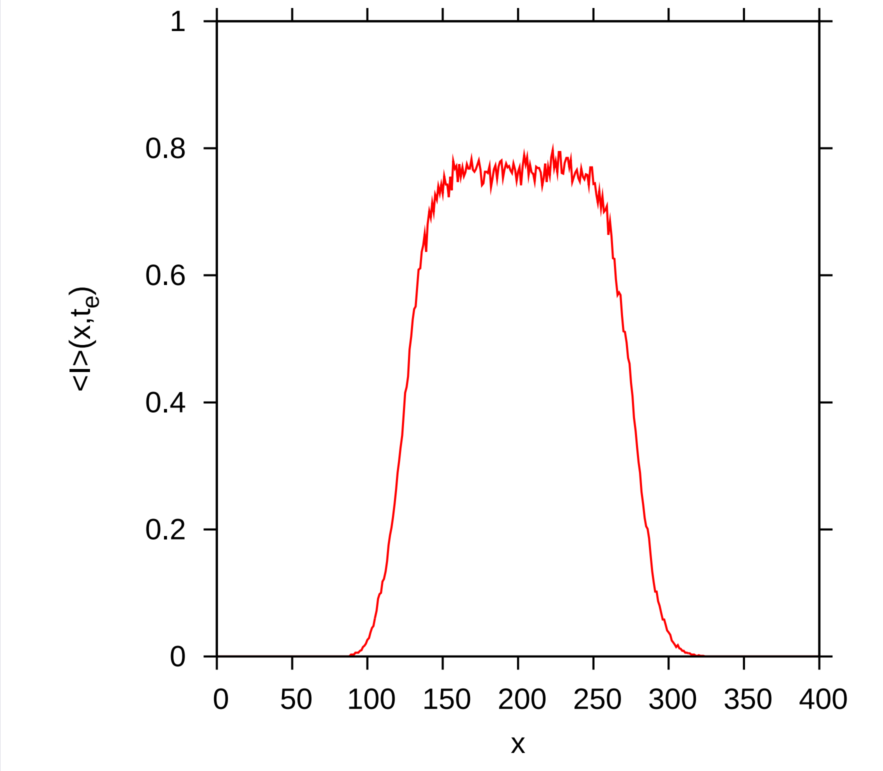
$$\langle I_i \rangle \approx u(\underline{x}_i, t)$$

Reaction-diffusion system in the SIS case

Kolmogorov-Fisher type equation

dynamics for local expectation values

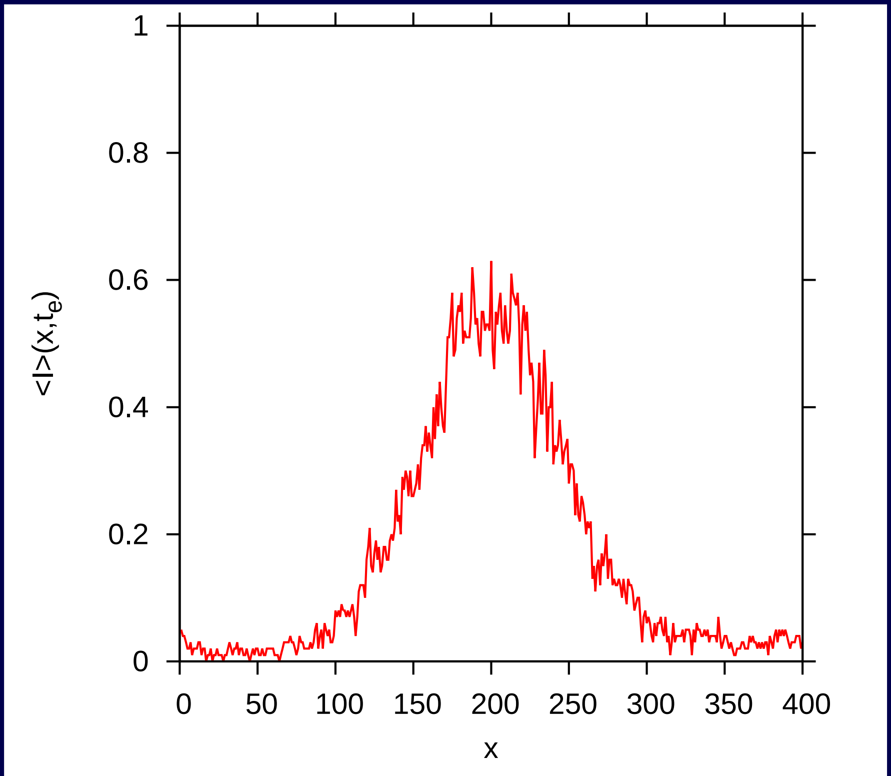
$$\frac{d}{dt}\langle I_i\rangle = \beta Q\langle I_i\rangle\left(1-\langle I_i\rangle\right) - \alpha\langle I_i\rangle + \beta(1-\langle I_i\rangle)\cdot\Delta\langle I_i\rangle$$



Reaction-super-diffusion system in the SIS case
Kolmogorov-Fisher type equation

dynamics for local expectation values

$$\frac{d}{dt}\langle I_i\rangle = \beta Q\langle I_i\rangle\left(1-\langle I_i\rangle\right)-\alpha\langle I_i\rangle+\beta(1-\langle I_i\rangle)\cdot\Delta^{\frac{\mu}{2}}\langle I_i\rangle$$



Power law jumps and power law waiting times

random walk

$$x_n = \sum_{i=1}^{n} \xi_i \quad , \quad t_n = \sum_{i=1}^{n} \tau_i$$

arrival probability $\eta(x,t)$ to have arrived at time t at the location x, jump prob. λ, waiting time prob. ψ,

$$\eta(x,t) = \int_{-\infty}^{\infty} \lambda(x-x') \int_0^t \psi(t-t')\eta(x',t')\,dt'\,dx' + \delta(x)\delta(t)$$

then the probability $p(x,t)$ to be at time t at location x via survival probability $\Psi(t-t') := 1 - \int_{t'}^{t} \psi(t'')\,dt''$

$$p(x,t) = \int_0^t \Psi(t-t')\eta(x,t')\,dt'$$

with convolutions in Fourier transform respectively in Laplace transform gives

$$\tilde{\bar{p}}(k,s) = \frac{1-\bar{\psi}(s)}{s} \cdot \frac{1}{1-\bar{\psi}(s)\cdot\tilde{\lambda}(k)}$$

Power law jumps and power law waiting times

Fourier-Laplace transform

$$\tilde{\bar{p}}(k,s) = \frac{1-\bar{\psi}(s)}{s} \cdot \frac{1}{1-\bar{\psi}(s)\cdot\tilde{\lambda}(k)}$$

with power law jump and waiting time probabilities for large arguments, hence for the respective transforms

$$\bar{\psi}(s) = 1 - s^{\nu} \quad \text{for} \quad s \to 0$$

$$\tilde{\lambda}(k) = 1 - |k|^{\mu} \quad \text{for} \quad |k| \to 0$$

with exponents μ and ν gives

$$\tilde{\bar{p}}(k,s) = \frac{s^{\nu-1}}{s^{\nu} + |k|^{\mu}}$$

the same result as in fractional calculus

Space-time fractional diffusion equation

generalized from $\nu = 1$ and $\mu = 2$

$$\frac{\partial^\nu}{\partial t^\nu} u(x,t) = \chi \frac{\partial^\mu}{\partial x^\mu} u(x,t)$$

with spatial Riesz fractional derivative via Fourier transform

$$\frac{\partial^\mu}{\partial x^\mu} e^{ikx} := -|k|^\mu \cdot e^{ikx}$$

and temporal Caputo fractional derivative via Laplace transform

$$\left(\int_0^\infty e^{-st} \left(\frac{\partial^\nu}{\partial t^\nu} f(t) \right) \, dt \right) := s^\nu \bar{f}(s) - s^{\nu-1} f(0)$$

we obtain as Fourier-Laplace transform

$$\bar{\tilde{u}}(k,s) = \tilde{u}(k, t_0 = 0) \, \frac{s^{\nu-1}}{s^\nu + \chi |k|^\mu}$$

Space-time fractional diffusion equation

solution via Laplace-back transform to Mittag-Leffler function

$$E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + 1)}$$

with

$$\int\limits_0^\infty e^{-st} E_\nu(ct^\nu)\, dt = \frac{s^{\nu-1}}{s^\nu - c}$$

and Fourier back-transform gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k, t_0) E_\nu\left(-\chi |k|^\mu (t - t_0)^\nu\right) \mathrm{e}^{ikx}\, dk$$

Space-time fractional diffusion equation

solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k,t_0) E_\nu \left(-\chi |k|^\mu (t-t_0)^\nu\right) \mathrm{e}^{ikx}\, dk$$

with initial condition in real space

$$u(x,t) = \int_{-\infty}^{\infty} u(y,t_0) G(x-y, t-t_0)\, dy$$

with Green's function

$$G(x-y, t-t_0) = \frac{1}{\chi^{\frac{1}{\mu}} (t-t_0)^{\frac{\nu}{\mu}}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\tilde{k}z} E_\nu(-|\tilde{k}|^\mu) d\tilde{k}$$

substituting $z := \frac{x-y}{\chi^{\frac{1}{\mu}}(t-t_0)^{\frac{\nu}{\mu}}}$ and $\tilde{k} := k\chi^{\frac{1}{\mu}}(t-t_0)^{\frac{\nu}{\mu}}$

Fast power law random numbers

waiting times with Mittag-Leffler random numbers with exponent ν

$$\tau = -\gamma_t \, ln(u) \left(\frac{sin(\nu\pi)}{tan(\nu\pi v)} - cos(\nu\pi) \right)^{\frac{1}{\nu}}$$

and jumps with Lévy stable random numbers with exponent μ

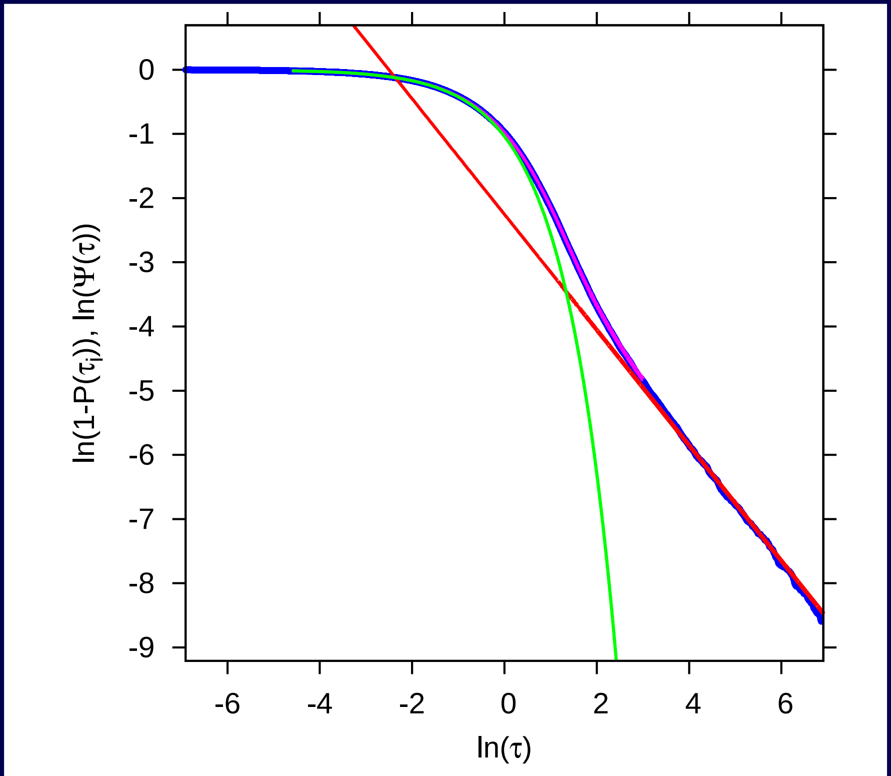
$$\xi = \gamma_x \left(\frac{-ln(u) \cdot cos(\phi)}{cos((1-\mu)\phi)} \right)^{1-\frac{1}{\mu}} \frac{sin(\nu\phi)}{cos(\phi)}$$

using uniformly distributed random numbers u and v on the unit interval and $\phi := \pi \left(v - \frac{1}{2} \right)$

Fast analysis via complementary distribution fct.

e.g. survival time distribution

$$\Psi(\tau) := \int_{\tau}^{\infty} \psi(\tilde{\tau})\, d\tilde{\tau} = 1 - \int_{0}^{\tau} \psi(\tilde{\tau})\, d\tilde{\tau} = 1 - P(\tau)$$



Human mobility in epidemiological context

"radiation model"
uses human densities to model mobility

=> power laws expected

twitter data in Thailand
a more direct surrogate for human mobility

=> power laws observable (?)

(as import into dengue models)

Human mobility in epidemiological models, radiation model

epidemiological models for Thai provinces $\underline{x}_i$ and populations N_i with import ϱ_i

$$\dot{\underline{X}}_i = \underline{f}_i(\underline{X}_i, \varrho_i)$$

now with explicit modelling of human mobility, hence $T_{j|i}$ the number of people moving from location $\underline{x}_i$ to $\underline{x}_j$ via total number of travelers from $\underline{x}_i$

$$T_i = \varepsilon N_i$$

assumed simplest proportional to population sizes and with elementary probability $p_{j|i}$ for a single person to move

$$p(T_{j|i}|T_i, p_{j|i}) = \binom{T_i}{T_{j|i}} p_{j|i}^{T_{j|i}} (1 - p_{j|i})^{T_i - T_{j|i}}$$

and $p_{j|i}$ still to be determined from detailed radiation model

Radiation model for $p_{j|i}$

define variable z as measuring "attractivity" of a location (or "absorption capacity" in physical radiation)

draw N_i times from probability distribution $p(z)$ obtaining maximal value z_i^{max}

and use extreme value statistics tools via cumulative distribution function

$$P(z) := \int_0^z p(\tilde{z})\, d\tilde{z}$$

hence for a fixed value z we have for N_i independent draws

$$\begin{aligned} p(z_i^{max} < z) &= p(z_{i,1} < z, z_{i,2} < z, ..., z_{i,N_i} < z) \\ &= p(z_{i,1} < z) \cdot p(z_{i,2} < z) \cdot ... \cdot p(z_{i,N_i} < z) \\ &= (P(z))^{N_i} \end{aligned}$$

Radiation model for $p_{j|i}$

define variable z as measuring "attractivity" of a location (or "absorption capacity" in physical radiation)

draw N_i times from probability distribution $p(z)$ obtaining maximal value z_i^{max}

and use extreme value statistics tools via cumulative distribution function

$$P(z) := \int_0^z p(\tilde{z})\, d\tilde{z}$$

hence for a fixed value z we have for N_i independent draws

$$p(z_i^{max} = z) \;=\; N_i \cdot P(z)^{N_i - 1} \cdot p(z)$$

with $p(z) = \frac{dP(z)}{dz}$, hence all expressed via cumulative distribution function $P(z)$

Description of "absorption" process

1) Emit at $\underline{x}_i$ a particle with attractivity value z_i^{max}. (The higher the population size N_i the higher z_i^{max} is likely to be).

2) Absorb not at any location $\underline{x}_k$ with attractivity $z_k^{max} < z_i^{max}$ in a circle with radius of distance $r_{ik} := ||\underline{x}_k - \underline{x}_i||$.

3) Only absorb at location $\underline{x}_j$ where the attractivity is $z_j^{max} > z_i^{max}$ at the minimal distance $r_{ij} := ||\underline{x}_j - \underline{x}_i||$.

Description of "absorption" process

1) The probability $p(z_i^{max} = z)$ of drawing z_i^{max} to be of value z at location $\underline{x}_i$ with population size N_i is given by

$$p(z_i^{max} = z) \; = \; N_i \cdot P(z)^{N_i - 1} \cdot \frac{dP(z)}{dz}$$

2) The probability $p(\bigwedge_{k:r_{ik}<r_{ij}} z_k^{max} < z)$ of not being absorbed at any location $\underline{x}_k$ due to $z_k^{max} < z_i^{max}$ is given by

$$p(\bigwedge_{k:r_{ik}<r_{ij}} z_k^{max} < z) \; = \; P(z)^{s_{ij}} \quad \text{with} \quad s_{ij} = \sum_{k:r_{ik}<r_{ij}} N_k$$

3) The probability $p(z_j^{max} > z)$ of being absorbed at nearest location $\underline{x}_j$ with $z_j^{max} > z_i^{max}$ is given by

$$p(z_j^{max} > z) \; = \; 1 - p(z_j^{max} < z) \; = \; 1 - P(z)^{N_j}$$

Description of "absorption" process

hence the elementary probability $p_{j|i}$ of being emitted in $\underline{x}_i$ and absorbed in $\underline{x}_j$ is given by the three contributions

$$p_{j|i} = \int_0^\infty p(z_i^{max} = z) \cdot p(\bigwedge_k z_k^{max} < z) \cdot p(z_j^{max} > z)\, dz$$

Description of "absorption" process

hence the elementary probability $p_{j|i}$ of being emitted in $\underline{x}_i$ and absorbed in $\underline{x}_j$ is given by the three contributions

$$
\begin{aligned}
p_{j|i} &= \int_0^\infty p(z_i^{max} = z) \cdot p(\bigwedge_k z_k^{max} < z) \cdot p(z_j^{max} > z) \, dz \\
&= \int_0^\infty N_i P(z)^{N_i - 1} p(z) \cdot P(z)^{s_{ij}} \cdot (1 - P(z)^{N_j}) \, dz \\
&= N_i \int_0^\infty \left(P(z)^{N_i + s_{ij} - 1} - P(z)^{N_i + s_{ij} + N_j - 1} \right) \frac{dP}{dz} \, dz \\
&= N_i \int_0^1 \left(P^{N_i + s_{ij} - 1} - P^{N_i + s_{ij} + N_j - 1} \right) dP \\
&= N_i \left(\frac{1}{N_i + s_{ij}} - \frac{1}{N_i + s_{ij} + N_j} \right) \\
&= \frac{N_i N_j}{(N_i + s_{ij})(N_i + N_j + s_{ij})}
\end{aligned}
$$

Mean connectivity $\langle T_{j|i} \rangle$

the mean connectivities between provinces $\langle T_{j|i} \rangle$ are now easily obtained from the binomial probabilities

$$p(T_{j|i}|T_i, p_{j|i}) = \binom{T_i}{T_{j|i}} p_{j|i}^{T_{j|i}} (1 - p_{j|i})^{T_i - T_{j|i}}$$

with elementary probabilities $p_{j|i}$ as calculated above

$$p_{j|i} = \frac{N_i N_j}{(N_i + s_{ij})(N_i + N_j + s_{ij})}$$

giving with $T_i = \varepsilon N_i$

$$\langle T_{j|i} \rangle = T_i \cdot p_{j|i} = \varepsilon N_i \cdot \frac{N_i N_j}{(N_i + s_{ij})(N_i + N_j + s_{ij})}$$

to be calculated for all $n = 76$ provinces in Thailand and its population sizes N_i and its distances $r_{ij} := ||\underline{x}_j - \underline{x}_i||$ from the coordinates $\underline{x}_i$ given in the World Geodetic System 1984 (WGS 84)

Plugging $\langle T_{j|i} \rangle$ into epidemiological models

single province models with import ϱ_i

$$\underline{\dot{X}}_i = \underline{f}_i(\underline{X}_i, \varrho_i)$$

become depending on infection levels and connectivities in other provinces

$$\underline{\dot{X}}_i = \underline{f}_i(\underline{X}_i, \varrho_i(\{\langle T_{j|i} \rangle, \underline{X}_j\}))$$

e.g. in SIS model with $S_i = (N_i - I_i)$

$$\dot{I}_i = \frac{\beta_i}{N_i}(I_i + \varrho_i N_i) S_i - \alpha_i I_i$$

becomes

$$\dot{I}_i = \frac{\beta_i}{N_i} I_i S_i - \alpha_i I_i + \sum_{j=1, j \neq i}^{n} \frac{\beta_j}{N_j} I_j \cdot T_{j|i} \frac{S_i}{N_i} + \sum_{j=1, j \neq i}^{n} \frac{\beta_i}{N_i} T_{i|j} \frac{I_j}{N_j} \cdot S_i$$

Plugging $\langle T_{j|i} \rangle$ into epidemiological models

single province models with import ϱ_i

$$\dot{\underline{X}}_i = \underline{f}_i(\underline{X}_i, \varrho_i)$$

become depending on infection levels and connectivities in other provinces

$$\dot{\underline{X}}_i = \underline{f}_i(\underline{X}_i, \varrho_i(\{\langle T_{j|i} \rangle, \underline{X}_j\}))$$

e.g. in SIS model with $S_i = (N_i - I_i)$

$$\dot{I}_i = \frac{\beta_i}{N_i}(I_i + \varrho_i N_i) S_i - \alpha_i I_i$$

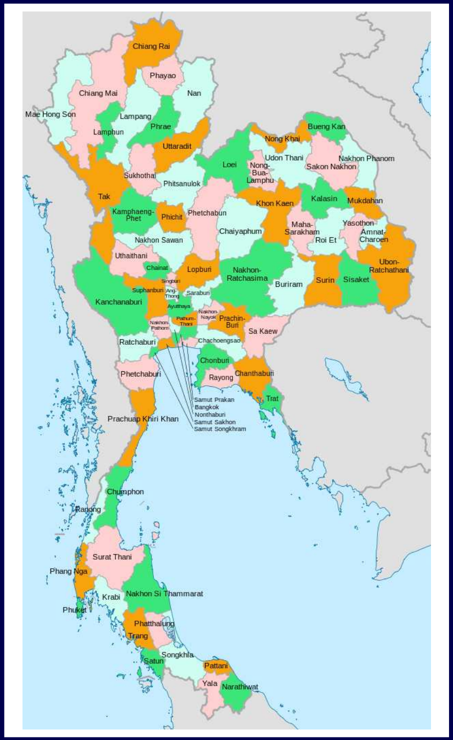
gives effective import

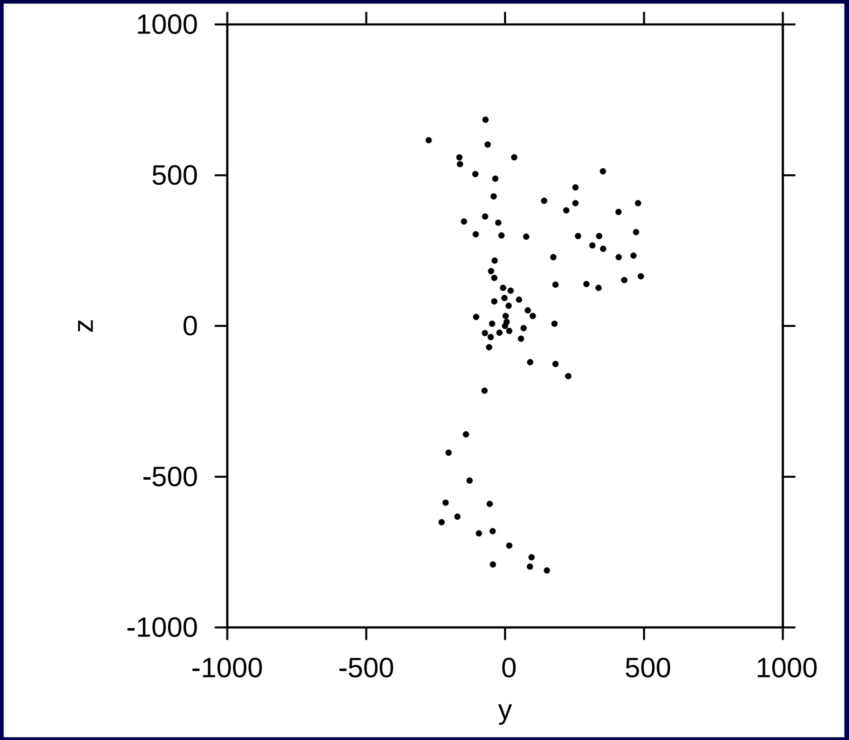
$$\varrho_i = \frac{1}{N_i} \sum_{j=1, j \neq i}^{n} \left(\frac{\beta_j}{\beta_i} T_{j|i} + T_{i|j} \right) \frac{I_j}{N_j}$$

New data via DENFREE:
34 years of DHF up to today, all provinces

Thailand with 77 changwats (provinces)

Implementation of the model for Thailand





locations $\underline{x}_i$ and connectivities $T_{j|i}$ calculated

power law statistics in preparation

One word of caution

the elementary probabilities are $p_{j|i}$ are in the binomial distribution normalized via $q_{j|i} := 1 - p_{j|i}$ being the probability of being emitted in i but not absorbed in j, and trivially $p_{j|i} + q_{j|i} = 1$

but along a finite network of n nodes the $p_{j|i}$ give

$$\sum_{j=1, j\neq i}^{n} p_{j|i} = \sum_{j=1, j\neq i}^{n} \frac{N_i N_j}{(N_i + s_{ij})(N_i + N_j + s_{ij})} = 1 - \frac{N_i}{N}$$

which is only approximately normalized in large networks with small nodes each via

$$N_i/N \to 0 \quad \text{for} \quad N_i \ll N$$

but a small probability N_i/N remains for particles to be emitted from i and not being absorbed in any other node, hence leaving any finite network